THE CALCULATION OF DOUBLE INTEGRALS

In this section we derive an easy method for calculating $\iint_{\Omega} f(x.y)\,dx\,dy$. where Ω is one of the regions . We begin, by considering

$$\iint_R f\,(x.y\,)dA \tag{1}$$

Where R is the rectangle $R = \{(x.y\,): a \le x \le b \text{ and } c \le y \le d\}$ (2)

If $z = f\,(x.y) \ge 0$ $for\ (x.y)\ in\ R$. then the double integral in (1) is the volume under the surface $z = f\,(x.y)$ and over the rectangle R in the xy –plane. We now calculate this volume by portioning the x- axis taking "slices" parallel to the yz- plane. This is illustrated in figure 1. We can approximate the volume by adding up the volumes of the various' 'slices" The face of each "slice" lies in the plane $x = x_i$. and the volume of the I th slice is approximately equal to the area of its face times its thickness Δx What is the area of the face? If x is fixed, then $z = f\,(x.y)$ can be thought by this curve, lying in the plane $x = x_i$. Thus the area of the ith face is the area bounded by this curve, the y-axis, and the lines y = c and y = d .If $f(x_i\,.y\,)$ is a continuous function of y, then the area of the ith face, denoted A_i is given by

$$A_i = \int_c^d f(x_i\,.y\,)\,dy$$

By treating x_i as a constant, we can compute A_i as an ordinary definite integral, where the variable is y Not, too that $A(x) = \int_c^d f(x\,.y\,)\,dy$ a function of x only and can therefore be integrated as in Chapter 5. Then the volume of the ith slice is approximated by

$$V_i \approx \left\{\int_c^d f(x_i\,.y\,)\,dy\right\}\Delta x$$

So that, adding up these "sub volumes" and taking the limit as approaches zero, we obtain

$$V = \int_a^b \left\{\int_c^d f(x_i\,.y\,)\,dy\right\}dx = \int_a^b A(x)\,dx \tag{3}$$

Definition 1: The expression in (3) is called a repeated integral or iterated we also have Integral. Since $V = \iint_R f(x.y)dA$.

We obtain

$$\iint_R f(x.y)dA = \int_a^b \left\{ \int_c^d f(x_i\,.y\,)\,dy \right\} dx \qquad (4)$$

REMARK 1. Usually we will write equation (4) without braces. We then have

$$\iint_R f(x.y)dA = \int_a^b \int_c^d f(x_i\,.y\,)\,dy\;\,dx \qquad (5)$$

REMARK 2. We should emphasize that the first integration in $\int_a^b \int_c^d f\,(x.y)\;\,dy\,dx$ is performed by treating x as a constant.

Similarly, if we instead begin partitioning the y- axis , we find that the area of the face of a "slice" lying in the plane $y = y_i$ is given by

$A_i = \int_a^b f(x.y_i)dx$.

Where now is an integral in the variable x. Thus as before,

$$V = \int_c^d \left\{ \int_a^b f(x_i\,.y\,)\,dx \right\} dy\,. \qquad (6)$$

And

$$\iint_R f(x.y)dA = \int_c^d \int_a^b f(x_i\,.y\,)\,dx\;\,dy \qquad (7)$$

EXAMPLE 1 Calculate the volume under the plane $z = x + 2y$ and over the rectangle

$R = \{(x.y): 1 \le x \le 2 \; and \; 3 \le y \le 5\}$

Solution. We calculated this volume in Example Using equation (5), we have

$$V = \iint_R f(x.2\,y)dA = \int_1^2 [\int_3^5 (x.2\,y)dy]\;\,dx$$

Similarly, using equation (7), we have

$$V = \int_{3}^{b} \{\int_{1}^{2} (x + 2y)dx\}dy = \int_{3}^{b} [(\frac{x^2}{2} + 2yx)|_{x=1}^{x=2}]\, dy$$

$$= \int_{3}^{b} \{2 + 4y) - (\frac{1}{2} + 2y)\}dy = \int_{3}^{5} (2y + \frac{3}{2})dy$$

$= \left(y^2 + \frac{3}{2}y\right)|_3^5 = 19$

EXAMPLE 2 Calculate the volume of the region beneath the surface $z = xy^2 + y^3$ and over the rectangle $R = \{(x.y): 0 \le x \le 2 \; and \; 1 \le y \le 3\}$

Solution. A computer- drawn sketch of this region is given in Figure 3. Using equation (5), we have

$$V = \int_0^2 \int_1^3 (xy^2 + y^3)dydx = \int_0^2 [(\frac{xy^3}{3} + \frac{y^4}{4})|_1^3]dx$$

$$= \int_0^2 \left[\left(9x + \frac{81}{4}\right) - \left(\frac{x}{3} + \frac{1}{4}\right)\right] dx = \int_0^2 (\frac{26}{3}x + 20)\, dx$$

$$= \left(\frac{13x^2}{3} + 20x\right)|_0^2 = \frac{52}{3} + 40 = \frac{172}{3}$$

You should verify that the same answer is obtained by using equation (7).

We now extend our results to more general regions Let

$$\Omega = \{(x.y): a \le x \le b \; and \; g_1(x) \le y \le g_2(x)\} \quad (8)$$

This region is sketched in Figure 4. We assume that for every x in [a, b],

$$g_1(x) \le g_2(x) \quad (9)$$

If we partition the x –axis as before, then we obtain slices lying in the planes $x = x_i$.

$A_i = \int_{g_1(x_i)}^{g_2(x_i)} f\,(x_i.y)dy$ $\qquad V_i = \{\int_{g_1(x_i)}^{g_2(x_i)} f\,(x_i.y)dy\}\Delta x$.

And

$$V = \iint_{\Omega} f(x.y)dA = \int_a^b \int_{g_1(x_i)}^{g_2(x_i)} f(x_i.y)dy\,dx \qquad (10)$$

Similarly, let

$\Omega = \{(x.y): h_1(y) \le x \le h_2(y) and\ c \le y \le d\}$

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x.y)dx\,dy \qquad (12)$$

We summarize these results in the following theorem:

Theorem1 :Let f be continuous over a region Ω given by equation (8) or (11)

$(i) If\, \Omega\ is\ of the\ \ form\ (8).where\ g_1\ and g_2\ are\ \ continuous\ .then$

$$\iint_{\Omega} f(x.y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x.y)\,dy\,dx$$

$(ii) If\, \Omega\ is\ of the\ \ form\ (11).where\ h_1\ and h_2\ are\ \ continuous\ .then$

$$\iint_{\Omega} f(x.y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x.y)\,dx\,dy$$

REMARK 1. We have not actually proved this theorem here but have merely indicated why it should be so A rigorous proof can be found in any advanced calculus text.

REMARK 2. Not that this theorem says nothing about volume. It can be used to calculate any double integral if the hypotheses of the theorem are satisfied and if each function being integrated has an antiderivative that can be written in terms of elementary functions

EXAMPLE 3 Find the volume of the solid under the surface $z = x^2 + y^2$ and lying above the region $\Omega = \{(x.y): 0 \le x \le 1\ and\ x^2 \le y \le \sqrt{x}\}$

Solution is sketched in Figure 7. We see $0 \le x \le 1\ and\ x^2 \le y \le \sqrt{x}$ that Then using (10), we have

$V = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)dy\,dx = \int_0^1 \left\{\left(x^2 y + \frac{y^3}{3}\right)|_{x^2}^{\sqrt{x}}\right\} dx$

$= \int_0^1 \{(x^2\sqrt{x} + \frac{(\sqrt{x})^3}{3}) - (x^2 \cdot x^2 + \frac{(x^2)^3}{3})\}dx$

$= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - x^4 - \frac{x^6}{3}\right)dx$

$= \left(\frac{2x^{\frac{7}{2}}}{7} + \frac{2x^{\frac{5}{2}}}{15} - \frac{x^5}{5} - \frac{x^7}{21}\right) = \frac{2}{7} + \frac{2}{15} - \frac{1}{5} - \frac{1}{21} = \frac{18}{105}$

We can calculate this integral in another way We note that x varies $x = y^2$ *and* $x = \sqrt{y}$ between the curves Then using (12), since $0 \le y \le 1$ *and* $y^2 \le x \le \sqrt{y}$ we have

$$V = \int_0^1 \int_{y^2}^{\sqrt{y}} (x^2 + y^2)dx\,dy.$$

Which is easily seen to be equal to 18/105

EXAMPLE 4 Let $f(x.y) = x^2 y$ Calculate the integral of f over the region bounded by the x-axis and the semicircle $x^2 + y^2 = 4. y \ge 0$

Solution. The region of integration is sketched in Figure 8. Using equation (8) , we see that $0 \le y \le \sqrt{4 - x^2} . -2 \le x \le 2.$ so that , integrating first with respect to y , we obtain

$$\iint_\Omega x^2 y\, dA = \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} x^2 y\, dydx$$

$$= \int_{-2}^{2} \left\{\frac{x^2 y^2}{2}|_0^{\sqrt{4-x^2}}\right\} dx = \int_{-2}^{2} \frac{x^2(4 - x^2)}{2} dx$$

$$= \int_{-2}^{2} (2x^2 - \frac{x^4}{2})dx = (\frac{2x^3}{3} - \frac{x^5}{10})|_{-2}^{2} = \frac{64}{15}$$

We can also use equation (11) and integrate first with respect to x. Then $-\sqrt{4 - y^2} \le x \le \sqrt{4 - y^2} . 0 \le y \le 2.$ *and*

$$V = \int_0^2 \int_{\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y\, dxdy = \int_0^2 \left(\frac{x^3 y}{3}\right)|_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}})\text{dy}$$

$$= \int_{-2}^{2} \frac{2}{3}(4-y^2)^{3/2} y dy = \frac{-2}{15}(4-y^2)^{\frac{5}{2}}|_0^2 = \frac{64}{15}$$

REVERSING THE ORDER OF INTEGRATION

EXAMPLE 5 *Evaluate* $\int_1^2 \int_1^{x2} \left(\frac{x}{y}\right) dydx$

Solution

$$\int_1^2 \int_1^{x2} \left(\frac{x}{y}\right) dydx = \int_1^2 \{xIn\, y\, |_1^{x2}\}\, dx = \int_1^2 xIn\, x^2\, dx = \int_1^2 2xIn\, \mathrm{x}\, dx$$

It is necessary to use integration by complete the problem. Setting $u = In\, x\ and\ dv = 2x\, dx.\ we\ have\ \ du = \left(\frac{1}{x}\right) dx\, .v = x^2\, .and$

$$\int_1^2 2x\, In\, x\, dx = x^2\, In\, x\, |_1^2 - \int_1^2 x\, dx = 4\, In\, 2 - \frac{x^2}{2}\ |_1^2 = 4\, In\, 2 - \frac{3}{2}$$

There is an easier way to calculate the double integral. We simply reverse the order of integration of integration is Figure 9. If we want to integrate first with respect to x, we note that we can describe the region by

$\Omega = \left\{(\, x.y\,) : \sqrt{y} \le x \le 2\ and\ 1 \le y \le 4\right\}$,Then

$$\int_1^2 \int_1^{x^2} dydx = \iint \frac{x}{y}_{\Omega} dA = \int_1^4 \int_{\sqrt{y}}^2 \frac{x}{y} dxdy = \int_1^4 \{\frac{x^2}{2y}\,|_{\sqrt{y}}^2\} dy$$

$$= \int_1^4 \left(\frac{2}{y} - \frac{1}{2}\right) dy = \left(2Iny - \frac{y}{2}\right)|_1^4 = 2In4 - \frac{3}{2}$$

$$= 4In - \frac{3}{2}$$

REMARK. Why is it legitimate to reverse the order of integration? There is a theorem † that asserts the following: Suppose that the region Ω can be written as

$$\Omega = \{(x.y\,) : a \le x \le b.\, g_1(x) \le y \le g_2(x)\} = \{(x.y\,) : c \le y \le d.\, h_1(y) \le x \le h_2(y)\}$$

Then if f is continuous on Ω .

$$\iint_{\Omega} f\,(x.y\,)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f\,(x.y\,)dy\,dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x.y\,)dx$$

EXAMPLE 6 : Compute $\int_0^2 \int_y^2 e^{x^2} dxdy$

Solution. We first observe that the double integral cannot be evaluated directly since it is impossible to find an antiderivative fore^{x^2} Instead, we reverse the order of integration Form Figure 10 we see that Ω can be written as $0 \le y \le x. 0 \le x \le 2. so$

$$\int_0^2 \int_y^2 e^{x^2} dxdy = \iint_{\Omega} e^{x^2}\, dA$$

$$= \int_0^2 \int_0^x e^{x^2} dydx = \int_0^2 \left(ye^{x^2}|_{y=0}^{y=x}\right)dx = \int_0^2 xe^{x^2}dx = \frac{1}{2}e^{x^2}|_0^2 = \frac{1}{2}(e^4 - 1)$$

EXAMPLE 7 : Revers the order of in the iterated integral $\int_0^1 \int_{\sqrt{y}}^2 f(x.y\,)dxdy$

Solution.. This region is divided into two sub regions Ω_1 *and* Ω_2 What happens if we integrate first with respect we see to y ?$In\ \Omega_1$. $0 \le y \le x^2$. $In\ \Omega_2$. $0 \le y \le 1. Thus$

$$\int_0^1 \int_{\sqrt{y}}^2 f(x.y\,)dxdy = \iint_{\Omega} f(x.y\,)dA = \iint_{\Omega_1} f(x.y\,)dA + \iint_{\Omega_2} f(x.y\,)dA$$

$$= \int_1^2 \int_0^{x^2} f(x.y)dydx + \int_1^2 \int_0^1 f(x.y)dy\,dx$$

EXAMPLE 8 : Find the volume in the first octant bounded by the three coordinate planes and the surface $z = 1/(1 + x + 3y)^3$

Solution. The solid here extends the infinite region $\{(x.y\,): 0 \le x \le \propto \quad and \quad 0 \le y \le \propto \} Thus$

$$V = \int_0^{\propto} \int_0^{\propto} \frac{1}{(1+3y)^3} dxdy = \int_0^{\propto} \lim_{N\to\propto} \left(-\frac{1}{2(1+x+3y)^2}|_0^N\right) dy$$

$$= \int_0^{\propto} \frac{1}{2(1+3y)^2} dy = \lim_{N\to\propto} \left(-\frac{1}{6(1+3y)}|_0^N\right) = \frac{1}{6}$$

Not that improper double integrals can be treated in the same way that we treat improper "single " integrals

EXAMPLE 9: Find the volume of the solid bounded by the three coordinate plane $2x + y + z = 2$

Solution. We have z = 2- 2x – y and this expression must be integrated over the region in the xy-plane bounded by the line 2x+ y = 2 (obtained when z = 0) and the x- and y – axes.

$$V = \int_0^1 \int_0^{2-2x} (2 - 2x - y)dydx = \int_0^1 \{\left(2y - 2xy - \frac{y^2}{2}\right)|_0^{2-2x}\}dx$$

$$= \int_0^1 \left\{2(2 - 2x) - 2x\,(2 - 2x) - \frac{(2-2x)^2}{2}\right\} dx$$

$$= \int_0^1 (2x^2 - 4x + 2)dx = \left(\frac{2x^3}{3} - 2x^2 - 2x\right)|_0^1 = \frac{2}{3}$$

EXAMPLE 10: Find the volume of the solid bounded by the circular paraboloid $x = y^2 + x^2$ and the plane $x = 1$

Solution. There are many ways to obtain its volume. Perhaps the best way to do so is to get another perspective on the picture. In Figure 14 we redraw Figure 13 with x as the vertical axis. The volume of the indicated element is $(1 - x)\,\Delta y\,\Delta z.$ $where\ x = y^2 + z^2$ We can take advantage of symmetry to write

$$V = 4\int_0^1 \int_0^{\sqrt{1-z^2}} (1 - x)dydx = 4\int_0^1 \int_0^{\sqrt{1-z^2}} (1 - y^2 - z^2)dydz$$

$$= 4\int_0^1 (y - \frac{y^3}{3} - yz^2)\ \ |_{y=0}^{\sqrt{1-z^2}} dz$$

$$= 4\int_0^1 \{(1 -)^{1/2} - \frac{(1-z^2)^{1/2}}{3} - z^2(1 - z^2)^{1/2}\}dz$$

We can integrate this by making the substitution $z = \sin\theta\ Then$

$$V = 4\int_0^{\frac{\pi}{2}} (\cos\theta - \frac{cos^3\theta}{3} - sin^2\theta\cos\theta)\cos\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} [cos\theta\,\underbrace{(1 - sin^2\theta)}_{=cos^2\theta} - \frac{cos^3\theta}{3}]\cos\theta\, d\theta$$

$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} cos^4\theta d\theta = \frac{2}{3}\int_0^{\frac{\pi}{2}} (1 + cos2\,\theta)^2\, d\theta$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} (1 + 2\cos 2\,\theta + cos^2\, 2\theta)d\theta$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} (1 + 2\cos 2\,\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta)d\theta = \frac{2}{3}\cdot\frac{3}{2}\cdot\frac{\pi}{2} = \frac{\pi}{2}$$

PROBLEMS : In problems 1-23 evaluate the given double integral.

1 · $\int_0^1 \int_0^2 xy^2 dxdy$

2 · $\int_{-1}^3 \int_2^4 (x^2 - y^3) dydx$

3 · $\int_2^3 \int_0^4 e^{(x-y)} dxdy$

4 · $\int_0^1 \int_{x^2}^{x} x^3 y\, dydx$

5 · $\int_2^4 \int_{1+y}^{2+3y} x - y^2 dxdy$

6 · $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_{sinx}^{cosx} (x + 2y) dy\, dx$

7 · $\int_0^3 \int_{\sqrt{9-y^2}}^{\sqrt{9-y^2}} xy^2 dxdy$

8 · $\int_1^2 \int_{y^5}^{3y^5} \frac{1}{x}\, dxdy$

9 · $\iint_\Omega (x^2 + y^2) dA. where\ \Omega = \{(\mathrm{x.y}): 1 \le \mathrm{x} \le 2 \text{ and} - 1 \le y \le 1\}$

10 · $\iint_\Omega 2xydA.\ where\ \Omega = \{(x.y): 0 \le x \le 4\ and\ 1 \le y \le 3\}$

11 · $\iint_\Omega (\mathrm{x} - \mathrm{y})^2\ \mathrm{dA. where}\ \Omega = \{(x.y): -2 \le x \le 2\ and\ 0 \le y \le 1\}$

12 · $\iint_\Omega \sin(2x + 3y) dA\ .where\ \ \Omega = \{(x.y): 0 \le x \le \pi/6\ and\ 0 \le y \le \pi/18\}$

13 · $\iint_\Omega xe^{(x^2+y)}\ dA\ .where\ \Omega$ is the region of Problem 10 ·

14 · $\iint_\Omega (x - y^2)$dA . where Ω is the region in the first quadrant bounded by the x − axis . the y − axis . and the unit circle

15 · $\iint_\Omega (x^2 + y) dA. where\ \Omega$ is the region of Problem 14

16 · $\iint_\Omega (x^3 + y^3) dA. where\ \Omega$ is the region of Problem 14

17 · $\iint_{\Omega}(x-2y)\mathrm{dA}$. where Ω is the triangular region in bounded by the lines $\mathrm{y} = \mathrm{x}. \mathrm{y} = 1 - \mathrm{x}$ and the $y-$ axis

18 · $\iint_{\Omega} xe^{(x+2y)}\ dA$. *where* Ω is the region of Problem 17 ·

19 · $\iint_{\Omega}(x^2+y)\mathrm{dA}$. where Ω is the region in the first quadrant between the parabolas $\mathrm{y} = x^2$ *and* $y = 1 - x^2$

20 · $\iint_{\Omega}\left(\frac{1}{\sqrt{y}}\right) dA$. *where* Ω is the region of Problem 19 ·

21 · $\iint_{\Omega}\left(\frac{\mathrm{y}}{\sqrt{x^2+y^2}}\right) dA$. *where* $\Omega = \{(\mathrm{x}.\mathrm{y}): 1 \le \mathrm{x} \le \mathrm{y} \text{ and } -1 \le y \le 2\}$

22 · $\iint_{\Omega}[e^{-y}/(1+x^2)]\ \mathrm{dA}$. where Ω is the first quadrant

23 · $\iint_{\Omega}(\mathrm{x}+y)e^{-(x+y)}\ \mathrm{dA}$. where Ω is the first quadrant

In problems 24-33, (a) sketch the region over which the integral is taken. Then (b) change the order of integration, and (c) evaluate the given integral.

24 · $\int_0^2\int_{-1}^3 dxdy$

25 · $\int_0^4\int_{-5}^8 (x+y)dydx$

26 · $\int_0^2\int_{-1}^3 \frac{y^3}{x^3}dxdy$

27 · $\int_0^1\int_0^x dydx$

28 · $\int_0^1\int_x^1 dydx$

29 · $\int_0^{\pi/2}\int_0^{cosy} y\ dxdy$

30 · $\int_0^2\int_0^{\sqrt{4-y^2}}(4-x^2)^{3/2}dxdy$

31 · $\int_0^1\int_{\sqrt{x}}^{\sqrt[3]{x}}(1+y^6)dydx$

32 · $\int_0^1\int_{\sqrt{y}}^1\sqrt{3+x^3}dxdy$

33 · $\int_0^{\infty}\int_x^{\infty}\frac{1}{(1+y^2)^{7/5}}dydx$

34 · *Show that if both integrals exist. then*

$$\int_0^x \int_{-5}^x f(x+y)dydx \;=\; \int_0^x \int_y^\infty f(x+y)dxdy$$

In problems 35 -44, find the volume of the given solid

35. The solid bounded by the plane $x+y+z=3$ *and the three coordinate planes*

36. The solid bounded by the planes $x=0. z=0.\ x+2y+z=6.$ *and* $-2y+z=6$

37 · The solid bounded by the cylinders $x^2+y^2=4$ *and* $y^2+z^2=4$

38 · The solid bounded by the cylinder $x^2+z^2=1$ *and the planes* $y=0$ *and* $y=2$

39 · T*he ellipsoid* $x^2+4y^2+9z^2=36$

40 · The solid bounded above by the sphere $x^2+y^2+z^2=9$ *and below by the plan* $z=\sqrt{5}$

41 · The solid bounded by the planes $y=0$. $y=x$. and the cylinder $x+z^2=2$

42 · The solid bounded by the parabolic cylinder $x=z^2$ *and planes* $y=1. y=5. z=1.$ *and* $x=0$

43 · The solid bounded by the parabolid $y=x^2+z^2$ *and the plan* $x+y=3$

44 · The solid bounded by the surface $z=e^{-(x+y)}$ *and the three coordinate plane*

45 · *Use a double integral to find the area of each of the regions bounded by the* $x-$*axis and the curves* $y=x^3+1$ *and* $y=3-x^2$

46 *Use a double integral to find the area of each of the regions bounded by the curves* $y=x^{\frac{1}{m}}$ *and* $y=x^{\frac{1}{n}}$. *where* m *and* n *are positive and* $n>m$

47. Le$f(x.y)=g(x)h(y)$. t where g *and* h are continuous Let Ω be the rectangle $\{(x.y): a\le x\le b \text{ and } c\le y\le d\}$ *Show that*

$$\iint_\Omega f(x.y)dA = \{\int_a^b g(x)dx\} \cdot \{\int_c^d h(y)dy\}$$

48. Sketch the solid whose volume is given by $V=\int_1^3\int_0^2(x+3y)dxdy$

49. Sketch the solid whose volume is given by $V=\int_0^1\int_{x^2}^{\sqrt{x}}\sqrt{x^2+y^2}\,dxdy$