THE DOUBLE INTEGRL

we began by calculating the area under a curve y = f (x) (and above the x – axis) for x in the interval [a, b]. We initially assumed that, on [a, b], f ≥ 0. We carry out a similar development by obtaining an expression which represents a volume in R^3

We being by considering an especially simple case. Let R denote the rectangle inR^2 given by

$$R = \{(x \,.\, y\,)\}: a \le x \ \le b \ \ and\ c \le y \ \le d\,\} \qquad (1)$$

This rectangle is sketched in Figure 1. Let z = f (x, y) be a continuous function that is nonnegative over R. That is, *f (x, y)* ≥ 0. for every *(x, y)* in R. We now ask: What is the volume "under " the surface z = f (x, y) and "over " the rectangle R?

Step 1. Form a regular partition (i.e., all subintervals have the same length) of the intervals [a, b]. and [c, d]:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\,. \qquad (2)$$

$$a = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = d\,. \qquad (3)$$

We then define

$$\Delta x = x_i - x_{i-1} = \frac{b-a}{n} \qquad (4)$$

$$\Delta y = y_i - y_{i-1} = \frac{b-a}{n}. \qquad (5)$$

And define the sub rectangles R_{ij} by

$$R_{ij} = \{(x.y\,): x_{i-1} \le x \le x_i \ \ and\ y_{i-1} \le y \le \ y_i\} \qquad (6)$$

For i = 1, 2, …., n and j =1, 2…, m. This is sketched in Figure 3. Note that there are nm sub rectangles R_{ij} covering the rectangle R.

Step 2. Estimate the volume under the surface and over each sub rectangle.

Let $(x_{i^*} \,.\, y_{j^*})$ be a point in R_{ij} Then the volume V_{ij} under the surface and overR_{ij} is approximated by

$$V_{ij} \approx f\big(x_{i^*} \,.\, y_{j^*}\big)\Delta x \Delta y = f\big(x_{i^*} \,.\, y_{j^*}\big)\Delta A. \qquad (7)$$

Where $\Delta A = \Delta x \Delta y$ is the area of R_{ij} The expression on the right –hand side of (7) is simply the volume of the parallelepiped (three-dimensional box) with base R_{ij} and height $(x_{i^*} \,.\, y_{j^*})$ This volume corresponds to the approximate area $A_i \approx f(x_{i^*})\Delta x_i$ the expression $f\left(x_{i^*} . y_{j^*}\right)\Delta A$ will not in general be equal to the volume under the surface S. But if are small the approximation will be a good one The difference between the actual V_{ij} and the approximate volume given in (7).

Step 3. Add up the approximate volumes to obtain an approximation to the total volume

The total volume is

$$V = V_{]]} + V_{12} + \cdots + V_{1m} + V_{21} + V_{22} + \cdots + V_{2m}$$

$$+ \cdots + V_{n1} + V_{n2} + \cdots + V_{nm} \qquad (8)$$

To simplify notation, we use the summation sign introduced in Section 5. 3. Sine we are summing over two variables I and j, we need two such:

$$V = \sum_{i=1}^{n}\sum_{j=1}^{m} V_{ij} \qquad (9)$$

The expression in (9) is called a double sum. If we the expression

In (9), we obtain the expression in (8). Then combining (7) and (9), we have

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_{i^*} . y_{j^*}\right)\Delta A \qquad (10)$$

Step 4. Take a limit as both Δx *and* Δy approach zero.

To indicate that this is happening, we define

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Geometrically, Δs is the length of a diagonal of the rectangle R_{ij} whose sides have lengthsΔx *and*Δy ,$\Delta s \Delta s \to 0$. the number of sub rectanglesR_{ij} increase without bound and the area of eachR_{ij} approaches zero.

This writing out is done by summing over j first and then over i. For example,

$$\sum_{i=1}^{3}\sum_{j=1}^{4} a_{ij} = \sum_{i=1}^{3}(a_{i1} + a_{i2} + a_{i3} + a_{i4})$$

$$= a_{11} + a_{12} + a_{13} + a_{14} + a_{21} + a_{22} + a_{23} + a_{24} + a_{31} + a_{32} + a_{33} + a_{34}$$

"true" volume over the approximation (10) gets better and better as $\Delta s \to 0$ which enables us to write

$$V = \lim_{\Delta s \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{i^*} . y_{j^*}) \Delta A \qquad (11)$$

EXAMPLE 1 Calculate volume under the plane and over the rectangle $R = \{(x.y): 1 \leq x \leq 2 \ and \ 3 \leq y \leq 5\}$

Solution. Step 1

For simplicity, we partition each of the intervals [1, 2]and [3, 5] into n subintervals of equal length (i.e. =n):

$1 = x_0 < x_1 < \cdots < x_n = 2$

$3 = y_0 < y_1 < \cdots < y_n = 5.$

Where

$$x_i = 1 + \frac{i}{n} . \ \Delta x = \frac{1}{n}$$

And

$$y_i = 3 + \frac{2i}{n} . \ \Delta y = \frac{2}{n}$$

Step 2

Then choosing $x_{i^*} = x_i \ and \ y_{j^*} = y_i$. we obtain

$$V_{ij} \approx f(x_{i^*} . y_{j^*}) \Delta A = (x_i + 2y_i) \Delta x \Delta y$$

$$= \left[\left(1 + \frac{i}{n}\right) + 2\left(3 + \frac{2i}{n}\right) \right] \frac{1}{n} \cdot \frac{2}{n}$$

$$= \left(7 + \frac{i}{n} + \frac{4j}{n}\right) \frac{2}{n^2}$$

Step 3

$$V_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} V \approx \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{14}{n^2} + \frac{2i}{n^3} + \frac{8j}{n^3}\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{14}{n^2} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{2i}{n^3} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{8j}{n^3}$$

(1) (2) (3)

It is not difficult to evaluate each of these double sums. There are n^2 terms in each sum. Since $\frac{14}{n^2}$ does not depend on I or j, we evaluate the sum(1) by simply adding up the term $\frac{14}{n^2}$ a total of n^2 times Thus

$$\sum_{i=1}^{n}\sum_{j=1}^{n} n^2\left(\frac{14}{n^2}\right) = 14$$

Next, if we set $i = 1\ in$ (3), then we have $\sum_{i=1}^{n}\frac{8j}{n^3}$ Similarly, setting $i = 2.3.4 \dots .n$ in (3) yields $\sum_{i=1}^{n}\frac{8j}{n^3}$ Thus in (3) we obtain the term $\sum_{i=1}^{n}\frac{8j}{n^3}$ times. But

$$\sum_{j=1}^{n}\frac{8j}{n^3} = \frac{8}{n^3}\sum_{j=1}^{n} j = \frac{8}{n^3}(1 + 2 + \cdots + n)$$

Equation 5.4. 14

$$\searrow = \frac{8}{n^3}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{4(n+1)}{n^2}$$

Thus

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{8j}{n^3} = n\left\{\sum_{j=1}^{n}\frac{8j}{n^3}\right\} = n\left[\frac{4(n+1)}{n^2}\right] = \frac{4(n+1)}{n}$$

To calculate (2), we use the same argument as in (3):

$$\sum_{j=1}^{n}\frac{2i}{n^3} = n\left(\frac{2i}{n^3}\right) = \frac{2i}{n^{2'}}$$

So that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{2i}{n^3}=\sum_{i=1}^{n}\frac{2i}{n^2}=\frac{2}{n^2}\sum_{i=1}^{n}i=\frac{2}{n^2}\left[\frac{n(n+1)}{2}\right]=\frac{n+1}{n}$$

Finally, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}V_{ij}\ \approx 14+\frac{4(n+1)}{n}+\frac{n+1}{n}$$

Step 4

Now as $\Delta s \to 0$. both and approach $0. so\ n=(b-a)/\Delta x \to \infty$Thus

$$V=\lim_{\Delta s\to 0}\sum_{i=1}^{n}\sum_{j=1}^{n}f\left(x_{i^*}\,.\,y_{j^*}\right)\Delta A=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}f\left(x_{i^*}\,.\,y_{j^*}\right)\Delta A$$

$$=\lim_{n\to\infty}\left[14+4\left(\frac{n+1}{n}\right)+\frac{n+1}{n}\right]=14+4+1=19$$

Definition 1 THE DOUBLE INTEGRAL Let $z=f(x.\ y)$ and let the rectangle R be given by (1). Let $\Delta A=\Delta x\Delta y$ Suppose that

$$\lim_{\Delta s\to 0}\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(x_{i^*}\,.\,y_{j^*}\right)\Delta A$$

exists and is independent of the way in which the points $(x_{i^*}\,.\,y_{j^*})$ are chosen. Then the double integral of f over R, written $\iint_R f(x.y)dA$. is defined by

$$\iint_R = f(x.y)dA=\lim_{\Delta s\to 0}\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(x_{i^*}\,.\,y_{j^*}\right)\Delta A \qquad (12)$$

If the limit in (12) exists, then the function f is said to be integrable over R.

We observe that this definition says nothing about volumes ,For example, if *f (x, y)*

Takes on negative values in R, then the limit in (12) will not represent the volume under the surface However, the limit in (12) may still exist, and in that case f will be integrable over R .

NOTE. $\iint_R f(x.y)dA$ Is a number, not a function This is analogous to the fact that the definite integral $\int_a^b f(x)\,dx$ is a number. We will not encounter indefinite double integrals in this book

Theorem 1 :Existence of the Double Integral Over a Rectangle If f is continuous on R, then f is intergrable over R .

We now turn to the question of defining double integrals over regions in $\mathbb{R}^2$ that are not rectangular. We will denote a region in $\mathbb{R}^2$ by Ω. In $g_1 . g_2 . h_1$ and h_2 denote continuous functions Amore general region Ω We assume that the region is bounded.This means that there is a number M such that for ever$(x.\ y) in\ \Omega . |(x.y)| = \sqrt{x^2+y^2} \leq M$ Since Ω is bounded, we can draw a rectangle R around it Let be defined over Ω We then define a new function F by

$$F(x.y) = \begin{cases} f(x.y). & for(x.y)in\ \Omega \\ 0. & for\ (x.y) in\ R\ \ but\ not\ in\ \Omega \end{cases} \qquad (13)$$

Definition 2 : INTEGRABILITY OVER A REGION Let f be defined for (x, y) in and let F be defined by (13) Then we write

$$\iint_\Omega f(x.y) = \iint_{\mathrm{R}} f(x.y)\ dA \qquad (14)$$

If the integral on the right exists. In this case we say that f is integrable over Ω

REMARK. If we divide R into nm sub rectangles, then we can see what is happening. for each sub rectangle R_{ij} that lies entirely $\Omega. f = f.$ So the volume of the "parallelepiped " a aboveR_{ij} is given by

$$V_{ij} \approx f(x_{i^*} . y_{j^*})\Delta x \Delta y = F(x_{i^*} . y_{j^*})\Delta x \Delta y$$

However, if R_{ij} is in R but not in Ω then $F = 0$. so

$$V_{ij} \approx F(x_{i^*} . y_{j^*})\Delta x \Delta y = 0$$

Finally, if R_{ij} is partly in Ω and outside of Ω . then there is no real problem since, as $\Delta s \to 0$. the sum of the volumes above these rectangles (along the boundary of Ω will approach zero ---un less the boundary Ω of is very complicated indeed. Thus we see that the limit of the sum of the volumes of the "parallepiceds" above R is the same as the volumes of the "parallepiceds" above Ω This should help explain the "reasonableness" of expression (14)

Theorem 2 Existence of the Double Integral Over a More General Region Let Ω be one of the regions depicted in Figure 7 where the functions $g_1 and g_2$ or h_1 and h_2 are continuous. Let F be defined by (13). If f is continuous over Ω. then f is integrable over Ω and its integral is given by (14).

REMARK 1. There are some regions Ω that are so complicated that there are functions continuous but not integrable over Ω We will not concern ourselves with such regions in this book.

REMARK 2. If f is nonnegative and integrable over Ω . then

$$\iint_{\Omega} f(x\,.\,y\,)dA$$

Is defined as the volume under the surface $z = f\ (x.y\ )$ and over the region Ω

REMARK 3. If the function f (x, y) =1 is integrable over Ω then

$$\iint_{\Omega} 1dA = \iint_{\Omega} dA \qquad (15)$$

equal to the area of the region Ω To see this, note that

$$V_{ij} \approx f\big(x_{i^*}\,.\,y_{j^*}\big)\Delta A = \Delta A\,.$$

So the double integral (15) is the limit of the sum of areas of rectangles in Ω

We close this section by stating five theorems about double integrals. Each one is analogous to a theorem a bout definite integrals.

Theorem 3 :If f is integrable over Ω then for any constant c, cf is integrable over Ω and

$$\iint_{\Omega} cf\ (x.\ y)dA = c\iint_{\Omega} f(x.y)dA \qquad (16)$$

Theorem 4: If f is integrable over Ω. Then f+ g is integrable over Ω and

$$\iint_{\Omega} [f\ (x.\ y) + g(x.y)]dA = \iint_{\Omega} f(x.y)dA + \iint_{\Omega} g(x.y)\ dA \qquad (17)$$

Theorem 5: If f is integrable over Ω_1 and Ω_2. where Ω_1 and Ω_2 have no points in common except perhaps those of their common boundary, then f is integrable over $\Omega = \Omega_1 \cup \Omega_2$. and

$$\iint_{\Omega} f(x.y)dA = \iint_{\Omega_1} f(x.y)dA + \iint_{\Omega_2} f(x.y)\ dA$$

Theorem 6 :If f and g are integrable over Ω *and* $f\,(x.\ y\,) \le g\,(x.y\,)$ for every (x , y) in Ω. *then* $\iint_{\Omega} f(x.y)dA \le \iint_{\Omega} g\,(x.y)dA$ (18)

Theorem 7: Let f be integrable over Suppose that there exist constants m and M such that

$$m \le \mathrm{f}\,(x.y\,) \le \mathrm{M} \quad (19)$$

For every (x, y) in Ω If A_{Ω} denotes the area of Ω . then

$$m_{A_{\Omega}} \le \iint_{\Omega} f(x.y)dA \le M_{A_{\Omega}} \quad (20)$$

EXAMPLE 2 Let Ω be the rectangle $\{(x.y\,): a \le x \le b \text{ and } c \le y \le d\}$ Find upper and lower bounds for

$$\iint_{\Omega} \sin(x - 3y^3)dA$$

Solution. Since $-1 \le \sin(x - 3y^3)dA \le 1.$ and since$A_{\Omega} = (b - a)(d - c)$. we have, using (20),

$$-(b - a)(d - c) \le \iint_{\Omega} \sin(x - 3y^3)dA \le (b - a)(d - c)$$

EXAMPLE 3 Let Ω be the disk $\{(x.y\,): x^2 + y^2 \le 1\}$ Find upper and lower bounds for

$$\iint_{\Omega} \frac{1}{1+x^2+y^2}dA$$

Solution .Since $0 \le x^2 + y^2 \le 1 \text{ in } \Omega\,.$ *we easily see that*

$$\frac{1}{2} \le \frac{1}{1 + x^2 + y^2} \le 1$$

Since the area of the disk is π we have

$$\frac{\pi}{2} \le \iint_{\Omega} \frac{1}{1 + x^2 + y^2}dA \le \pi$$

PROBLEMS

In problems 1- 8, let denote the rectangle $\{(x.y): 0 \le x \le 3 \; and \; 1 \le y \le 2\}$Use the technique employed in Example 1to calculate the given double integral. Use Theorem 3 and /or 4 where appropriate.

1 · $\iint_\Omega (2x+3y)dA$

2 · $\iint_\Omega (x-y)dA$

3 · $\iint_\Omega (y-x)dA$

4 · $\iint_\Omega (ax+by+c)dA$

5 · $\iint_\Omega (x^2+y^2)dA$

6 · $\iint_\Omega (x^2-y^2)dA$

7 · $\iint_\Omega (2x^2+3y^2)dA$

8 · $\iint_\Omega (ax^2-by)dA$

In problems 9-14 let Ω denote the rectangle $\{(x.y): 0 \le x \le 3 \; and -2 \le y \le 3\}$ Calculate the double integral.

9 · $\iint_\Omega (x+y)dA$

10 · $\iint_\Omega (3x-y)dA$

11 · $\iint_\Omega (y-2x)dA$

12 · $\iint_\Omega (x^2+2y^2)dA$

13 · $\iint_\Omega (x^2-y^2)dA$

14 · $\iint_\Omega (3x^2-5y^2)dA$

15 · $\iint_\Omega (x^2y^2+xy)dA. where\ \Omega$ is the $rectangle\ \{(x.y): 0 \le x \le 3 \; and \; 1 \le y \le 2\}$

16 · $\iint_\Omega e^{-(}x^2-y^2)\, dA\ .where\ \Omega\ is\ the\ \ disk\ x^2+y^2 \le 4$

17 ·· $\iint_\Omega [(x-y)/(4-x^2-y^2)]dA. where\ \Omega\ is\ the\ \ disk\ x^2+y^2 \le 1$

18 · $\iint_\Omega \cos(\sqrt{x}-\sqrt{y})\, dA. where\ \Omega\ is\ the\ region\ of\ Problem\ 17$

19 · $\iint_\Omega In\ (1+x+y)dA. wheere\ \Omega\ is\ the\ region\ bounded\ by\ the\ lines\ y=x.y$
$=1-x\ and\ the\ x-axis$