CONSTRAINED MAXIMA AND MINIMA

LAGRANGE MULTIPLIERS

In the previous section we saw how to find the maximum and minimum of a function of two variables by taking gradients and applying a second derivative test. It often happens that there are side conditions (or conditions) attached to a problem. For example, we have been asked to find the shortest distance from a point (x_0 , y_0) to a line $y = mx + b$. We could write this problem as follows :

Minimize the function : $z = \sqrt{(x - x_0)^2 + (y - y_0)^2}$

Subject to the constraint : $y - mx - b = 0$.

As another example, suppose that a region of every point of the region. Then if the sphere is given and if we wish to find the hottest point on the sphere, we have the following problem:

Minimize : $w = T(x, y, z)$

Subject to the constraint : $x^2 + y^2 + z^2 - r^2 = 0.$

We now generalize these two examples. Let f and g be functions of two variables. Then we can formulate a constrained minimization (or minimization) problem as follows:

Minimize : $z = f\ (x.y)$ (1)

Subject to the constraint : $g(x , y) = 0$ (2)

If f and g are functions of three variables, we have the following problem:

Minimize (or minimize) : $w = f\ (x, y , z\ )$ (3)

Subject to the constraint : $g(x , y , z) = 0.$ (4)

We now develop a method for dealing with problems of the type (1),) (2), or (3), (4). Let C be a curve in R^2or R^3 given parametrically by the differentiable function F(t) That is, C is given by

$F(t) = x(t)i + y(t)j$ (in R^2)

Or

$F(t) = x(t)i + y(t)j + z(t)k$ (in R^3)

Let f(x) denote the function (of two or three variables) that is to be maximized.

Theorem 1. Suppose that f is differentiable at a point x_0 and that among all points on a curve C , f takes its maximum(or minimum) value at x_0 . Then $\nabla f(x_0)$ is orthogonal to C at x_0 That is since F' (t) is tangent to C, if $x_0 = F(t_0)$, then

$$\nabla f(x_0) \, . F(t_0) = 0 \qquad (5)$$

Proof. For x on C, x = F(t) , so that the composite function f (F(t)) has a maximum (or minimum) at t_0 Therefore its derivative at t_0 is 0 . By the chain rule,

$$\frac{d}{dt} f\left(F(t)\right) = \nabla f\left(F(t)\right) . F'(t),$$

And at t_0

$$0 = \nabla f\left(F(t)\right) . F'(t_0) = \nabla f(x_0) \, . \, F'(t_0) \, ,$$

And the theorem is proved.

For f as a function of two variables, the result of Theorem 1 is illustrated

We can use Theorem 1 to make an interesting observation. Suppose that, subject to the constraint g (x, y) = 0, f takes its maximum (or minimum) at the Point The equation g (x, y) =0 determines a curve C in the xy – plane, and by Theorem 1, is orthogonal to Cat But form Section 18.7, is also orthogonal to C Thus we see that

$$\nabla g(x_0 . y_0) and \; \nabla f \; (x_0 . y_0) are \;\; parallel$$

Hence there is a number λ such that

$$\nabla f(x_0 . y_0) = \lambda \nabla g \; (x_0 . y_0)$$

We can extend this observation to following rule, which applies equally well to functions of three or more variables:

$$If. supject \; to \; the \; constraint \;\; g(x) = 0. f \; takes \; its \; maximum \; (or \; minmum) value \; at \; a \; point x_0 . then \;\; thereis \; a \; number \; \lambda \; such \;\; that$$

$$\nabla f(x_0) = \lambda \nabla g \; (x_0) \qquad (6)$$

The number is called a Lagrange multiplier We will illustrate the Lagrange multi-plier technique with a number of examples

EXAMPLE 1 : Find the maximum values of f (x, y)= xy^2 subject to the condition

$x^2 + y^2 = 1$

Solution .We have g (x, y)= $x^2 + y^2 - 1 = 0$ Then

$\nabla f = y^2 i + 2xy^2 j$ and $\nabla g = 2xi + 2yj$

At a maximizing or maximizing point we have

$\nabla f = \lambda \, \nabla g$ .Or

$y^2 i + 2xyj = 2x\lambda i + 2y\lambda j$.

Which leads to the equations

$y^2 = 2x\lambda$

$2x\lambda = 2y\lambda$

Multiplying the first equation by y and the second by x, we obtain

$y^3 = 2xy\lambda$

$2x^2 y = 2xy\lambda$.

Or

$y^3 = 2x^2 y$

But$x^2 = 1 - y^2$. so

$y^3 = 2(1 - y^2)y = 2y - 2y^3$.

Or

$3y^3 = 2y$

The solutions to this last equation are y =0 and y = $\pm\sqrt{\frac{2}{3}}$ This leads to the six points

$(1.0)(-1.0).\left(\frac{1}{\sqrt{3}} \; .\sqrt{\frac{2}{3}}\right).\left(-\frac{1}{\sqrt{3}} \; .\sqrt{\frac{2}{3}}\right).\left(\frac{1}{\sqrt{3}} - \sqrt{\frac{2}{3}}\right).\left(-\frac{1}{\sqrt{3}} - \sqrt{\frac{2}{3}}\right)$

Evaluating $f(x . y) = xy^2$ at these points , we have

$$f(1 . 0) = f(-1 \ .0) = 0 \ . f\left(\frac{1}{\sqrt{3}} \ . \sqrt{\frac{2}{3}}\right) = f\left(\frac{1}{\sqrt{3}} . -\sqrt{\frac{2}{3}}\right) = \frac{2}{3\sqrt{3}} .$$

And

$$f\left(-\frac{1}{\sqrt{3}} \ . \sqrt{\frac{2}{3}}\right) = f\left(-\frac{1}{\sqrt{3}} . -\sqrt{\frac{2}{3}}\right) = -\frac{2}{3\sqrt{3}}$$

Thus the maximum value of f is$\frac{2}{3\sqrt{3}}$ and the minimum value of f is $-\frac{2}{3\sqrt{3}}$ Note that there is neither a maximum nor a minimum at $(1.0\ )$and (-1.0) even though$\nabla f(1.0) = 0\nabla g(1 . 0)$ and$\nabla f(-1.0) = 0\nabla g(-1.0) = 0$

EXAMPLE 2 Find the points sphere $x^2 + y^2 + z^2 = 1$ closest to and farthest from the point (1, 2, 3)

Solution. We wish to minimizef $(x . y . z) = (x-1)^2 + (y-2)^2 + (z-3)^2$ and maximize subject to $g(x. y . z) = x^2 + y^2 + z^2 - 1 = 0$ We have $\nabla f(x . y . z) = 2(x-1)i + 2(y-2)j + 2(z-3)k.$ and $\nabla g(x . y . z) = 2xi + 2yj + 2zk$ Condition
(6) implies that at amaximizing point $. \nabla f = \lambda \nabla g$. so

$2(x-1) = 2x\lambda$

$2(y-2) = 2x\lambda$

$2(z-3) = 2x\lambda$

If $\lambda \neq 1$. then we find that

$x - 1 = x\lambda$. or $x - x\lambda = 1$. or $x(1-\lambda) = 1$.

And

$$x = \frac{1}{1-\lambda}$$

Similarly, we obtain

$$y = \frac{2}{1-\lambda} \quad \text{and} \quad z = \frac{3}{1-\lambda}$$

Then

$$1 = x^2 + y^2 + z^2 = \frac{1}{(1-\lambda)^2} \quad (1^2 + 2^2 + 3^2) = \frac{14}{(1-\lambda)^{2'}}$$

So that $(1-\lambda)^2 = 14. (1-\lambda) = \pm\sqrt{14}$.

And $\lambda = 1 \pm \sqrt{14}$

$$\text{If } \lambda = 1 + \sqrt{14} \text{ then } (x.y.z) = \left(-\frac{1}{\sqrt{14}}.-\frac{2}{\sqrt{14}}.-\frac{3}{\sqrt{14}}\right) = 1 - \sqrt{14}$$

$$\text{then } (x.y.z) = \left(\frac{\frac{1}{\overline{\sqrt{14.2}}}}{\sqrt{14}}.3\sqrt{14}\right)$$

Finally. evaluation shows us that f is maximized at

$(-1/\sqrt{14}.-2/\sqrt{14}\ .-3/\sqrt{14}\ )$and that f is minimized at $(1/\sqrt{14}.2/\sqrt{14}\ .3/\sqrt{14}\ )$

We note that we can use Lagrange multipliers in R^2 there are two or more constraint equations. Suppose, for example, we wish to maximize (or minimize) $w = f(x.y.z)$ subject to constraints

$$g(x.y.z) = 0 \qquad (7)$$

And

$$h(x.y.z) = 0 \qquad (8)$$

Each of the equations (7) and (8) represents a surface in R^3 and their intersection forms a curve in R^3 By an argument very similar to the one we used earlier (but applied in R^3 instead of R^2) we find that if f is maximized (or minimized) at $(x_0.y_0.z_0)$ then $\nabla f(x_0.y_0.z_0)$ in the plane determined by$\nabla g(x_0.y_0.z_0)$ and $\nabla h(x_0.y_0.z_0)$ thus there are numbers λ and μ such that

$$\nabla f(x_0.y_0.z_0) = \lambda \nabla g(x_0.y_0.z_0) + \mu \nabla h(x_0.y_0.z_0) \qquad (9)$$

EXAMPLE 3 :Find the maximum value of$w = xyz$ among all points $(x.y.z)$ lying on the of intersection of planes $x + y + z = 30$ and $x + y - z = 0$

Solution Setting $f(x.y.z) = xyz\ .g(x.y.z) = x + y + z - 30$ .and

$$h(x.y.z) = x + y - z. \text{we obtain}$$

$\nabla f = yzi + xzj + xyk$

$\nabla g = i + j + k$

$\nabla h = i + j + k$.

And using equation (9) to obtain the maximum, we obtain the equations

$yz = \lambda + \mu$

$xz = \lambda + \mu$

$xy = \lambda - \mu$

Multiplying the three equations by x, y, and respectively, we find that

$xyz = (\lambda + \mu)x$

$xyz = (\lambda + \mu)y$

$xyz = (\lambda - \mu)z$

If $\lambda + \mu = 0$. then y z = 0 and xyz = 0, which is not a maximum value since xyz can be positive (for example,$x = 8. y = 7. z = 15$ is in the constraint set and xyz= 840) Thus we can divide the first two equations by $\lambda + \mu$ to find that.

x = y

Since $x + y - z = 0$ we have $2x - z = 0.$ or $z = 2x$ But then

$$30 = x + y + z = x + x + 2x = 4x\,.$$

Or

$x = \dfrac{15}{2}$

Then

$y = \dfrac{15}{2}$. $z = 15$.

And the maximum value of xyz occurs at ($\frac{15}{2}$. $\frac{15}{2}$ .15) and is equle to $\left(\frac{15}{2}\right)\left(\frac{15}{2}\right)15 = 843\ \frac{3}{4}$

PROBLEMS

1 · Use Lagrange multiers to find the minimum distance from the point $(1.2)$to the line $2x + 3y = 5$

2 · Use Lagrange multiers to find the minimum distance from the point $(3.-2)$to theline $y = 2 - x$

3 · Use Lagrange multiers to find the minimum distance from the point $(1.-1.2)$to the plane $x + y - z = 3$

4 · Use Lagrange multiers to find the minimum distance from the point $(3.0.1)$to the plane $2x - y + 4z = 5$

5 · Use Lagrange multiers to find the minimum distance from the plane $ax + by + cz = d$ to the origin

6 · Find the maximum and minimum values of $x^2 + y^2$ subject tothe condition $x^3 + y^3 = 6xy$

7 · *Find the maximum and minimum values of* $2x^2 + xy + y^2 - 2y$ *subject tothe condition* $y = 2x - 1$

8 · *Find the maximum and minimum values of* $x^2 + y^2 + z^2$ *subject tothe condition* $z^2 = x^2 - 1$

9 · *Find the maximum and minimum values of* $x^3 + y^3 + z^3$ *if* $(x.y.z)$*lies on the sphere* $x^2 + y^2 + z^2 = 4$

10 · *Find the maximum* ***and minimum values of*** $\boldsymbol{x + y + z}$ ***if*** $\boldsymbol{(x.y.z)}$ ***lies on the sphere*** $\boldsymbol{x^2 + y^2 + z^2 = 1}$

11 ***Find the maximum and minimum values of*** $\boldsymbol{xyz}$ ***if*** $\boldsymbol{(x.y.z)}$***is on the ellipsoid*** $\boldsymbol{x^2 + (y^2/4) + (z^2/9) = 1}$

12 · $\boldsymbol{Solve\ Problem\ 11\ if\ (x.y.z) is\ on\ the\ ellipsoid\ \left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)+\left(\frac{z^2}{c^2}\right)=1}$

$*$ **13 ·** $\boldsymbol{Minimize\ the\ function\ x^2+y^2+z^2\ \ for\ (x.y.z) on\ the\ planes\ 3x-y+z=6\ \text{and}\ x+2y+2z=2}$

14 · Find the minimum values of $x^3+y^3+z^3$ for $(x.y.z)$is on the planes $x+y-z=3$

15 · Find the maximum and minimum distance from the origin to a point on the ellipsoid $\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)=1$

16 · Find the maximum and minimum distance from the origin to a point on ellipsoid $\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)+\left(\frac{z^2}{c^2}\right)=1$