HIGER – ORDER PARTIAL DERIVATIVES

We have seen that if y = f(x), then

$$y' = \frac{df}{dx} \quad \text{and} \quad y'' = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

That is ,the second derivative of f is the derivative of the first derivative of f. , if z = f (x, y), then we can differentiate each of the two "first" partial derivatives∂ fl ∂ x and∂ fl ∂y with respect to both x and y to obtain four second partial derivatives as follows :

Definition 1: SECOND PARTIAL DERIVATIVES

(i) Differentiate twice with respect to x :

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad (1)$$

(ii) Differentiate first with respect to x and then with respect to y :

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial y\,\partial x} = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad (2)$$

(iii) Differentiate first with respect to y and then with respect to x :

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial x\,\partial y} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad (3)$$

(iv) Differentiate twice with respect to y :

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \quad (4)$$

REMARK1.Thederivatives $\partial^2 f l \partial$ x ∂y and $\partial^2 f \partial$ y ∂x arecalled the mixed second partialsl.

REMARK2. It is much easier to denote the second partials byf_{xx}, f_{xx}, f_{yx} andf_{yy}, We

Will there f are use this notation for the remainder of this section. Note that the symbol f_{xy} indicates that we differentiate first with respect to y .

EXAMPLE 1 Let $z = f(x,y) = x^3y^2 - xy^5$ .Calculate the four second partial derivatives .

Solution. We have $f_x = 3x^2y^2 - y^5$ and $f_y = 2x^3y - 5xy^4$.

(a) $f_{xx} = \frac{\partial}{\partial x}(f_x) = 6xy^2$

(b) $f_{xy} = \frac{\partial}{\partial y}(f_x) = 6x^2 y - 5y^2$

(c) $f_{xy} = \frac{\partial}{\partial x}(f_y) = 6x^2 y - 5y^2$

(d) $f_{yy} = \frac{\partial}{\partial y}(f_y) = 2x^3 - 20xy^3$

In Example 1 we saw that$f_{xy} = f_{yx}$ This result is no accident ,as we see by the following theorem whose proof can be found in any intermediate calculus text .t

Theorem 1 : Suppose that f, f_x, f_y, f_{xy} and f_{yx} are all continuous at (x_0 , y_0) Then

$$f_{xy}(x_0 , y_0) = f_{yx} (x_0 , y_0) \qquad (5)$$

This result is often referred to as the equality of mixed partials ≠0

The definition of second partial derivatives and the theorem on the equality of mixed partials are easily extended to functions of three variables. If w =(x ,y , z), then we have the nine second partial derivatives (assuming that they exist) :

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} , \frac{\partial^2 f}{\partial y \ \partial x} = f_{xy} , \quad \frac{\partial^2 f}{\partial z \ \partial x} = f_{xz} ,$$

$$\frac{\partial^2 f}{\partial x \ \partial y} = f_{yx} , \frac{\partial^2 f}{\partial y^2} = f_{yy} , \quad \frac{\partial^2 f}{\partial z \ \partial y} = f_{yz}$$

$$\frac{\partial^2 f}{\partial x \ \partial z} = f_{zx} , \frac{\partial^2 f}{\partial y \ \partial z} = f_{zy} , \quad \frac{\partial^2 f}{\partial z^2} = f_{zz}.$$

Theorem2 If f, f_x, f_y, f_z and f_{yx} and all six mixed partial are continuous at a point (x_0 , y_0 , z_0)then at a point This theorem was first stated by Euler in a 1734 paper devoted to a problem in hydrodynamics

$$f_{xy} = f_{yx} , \quad f_{xz} = f_{zx} , \quad f_{yz} = f_{zy} .$$

EXAMPLE 2 : Let $f(x,y,z) = xy^3 - zx^5 + x^2yz$ be a function , Calculate all for nine second partial derivatives and show that all three pairs of mixed partials are equal

Solution : We have

$f_x = y^3 - 5zx^4 + 2xyz$,

$f_y = 3xy^2 + x^2z$,

and

$f_z = -x^5 + x^2y$

Then

$f_{xx} = -20zx^3 + 2yz$, $f_{yy} = 6xy$, $f_{zz} = 0$,

$$f_{xy} = \frac{\partial}{\partial y}(y^3 - 5zx^4 + 2xyz) = 3y^2 + 2xz,$$

$$f_{yx} = \frac{\partial}{\partial x}(3xy^2 + x^2z) = 3y^2 + 2xz,$$

$$f_{xz} = \frac{\partial}{\partial z}(y^3 - 5zx^4 + 2xyz) = -5x^4 + 2xy,$$

$$f_{zx} = \frac{\partial}{\partial x}(-x^5 + x^2y) = -5x^4 + 2xy,$$

$$f_{yz} = \frac{\partial}{\partial z}(3xy^2 + x^2z) = x^2,$$

$$f_{zy} = \frac{\partial}{\partial y}(-x^5 + x^2y) = x^2$$

We conclude this section by pointing out that we can easily define partial derivatives of orders higher than two . For example,

$$f_{zyx} = \frac{\partial^3 f}{\partial x\,\partial y\,\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\,\partial z}\right) = \frac{\partial}{\partial x}(f_{zy})$$

EXAMPLE 3 Calculate and for the function of Example 2 .

Solution We easily obtain the three third partial derivatives:

$$f_{xxx} = \frac{\partial}{\partial x}(f_{xx}) = \frac{\partial}{\partial x}(20zx^3 + 2yz) = -60zx^2$$

$$f_{zzy} = \frac{\partial}{\partial y}(f_{xz}) = \frac{\partial}{\partial y}(5x^4 + 2yz) = 2x$$

$$f_{yxz} = \frac{\partial}{\partial z}\left(f_{yx}\right) = \frac{\partial}{\partial z}(3y^2 + 2xz) = 2x$$

Note that $f_{xzy} = f_{yxz}$This again is no accident and follows from the generalization of Theorem 2 to mixed third partial derivatives. Finally, the fourth partial derivative $f_{yxzx}$is given by

$$f_{yxzx} = \frac{\partial}{\partial x}\left(f_{yxz}\right) = \frac{\partial}{\partial x}(2x) = 2.$$

PROBLEMS

In problems 1-12 , calculate the four second partial derivatives and show that the mixed partials are equal .

1 . $f(x, y) = x^2 y$.

. $f(x, y) = xy^2 y$.2

3 . $f(x, y) = 3e^{xy3}$

4. $f(x, y) = \sin(x^2 + y^3)$

5 . $f(x, y) = \frac{4x}{y^5}$

6 . $f(x, y) = e^y \tan_x$.

7 . $f(x, y) = \text{In}(x^3 y^5 - 2)$

8 . $f(x, y) = \sqrt{xy + 2y^3}$

9. $f(x, y) = (x + 5y \sin x)$اكتبالمعادلةهنا.

10 . $f(x, y) = \sinh(2x - y)$

11 . $f(x, y) = \sin^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$

12 . $f(x, y) = \sec xy$

In Problems 13 -21 , calculate the nine second partial derivatives and show that the three pairs of mixed partials are equal

13 . $f(x, y, z) = xyz$

14 . $f(x, y, z) = x^2y^3z^4$

15 . $f(x, y, z) = \frac{x+y}{z}$

16 . $f(x, y, z) = \sin(x + 2y + z^2)$

17. $f(x, y, z) = \tan^{-1}\frac{xz}{y}$

18 . $f(x, y, z) = \cos xyz$

19 . $f(x, y, z) = e^{3xy} \cos z$

20 . $f(x, y, z) = \ln(xy + z)$

21 . $f(x, y, z) = \cosh\sqrt{x + yz}$

22 . How many third partial derivatives are there for a function of (a) two variables; (b) three variables?

23 . How many fourth partial derivatives are there for a function of (a) two variables; (b) three variables?

24 . How many nth partial derivatives are there for a function of (a) two variables; (b) three variables ?

In Problems 25 -30 , calculate the given partial derivative

25 . $f(x, y) = x^2y^3 + 2y$: f_{xyx}

26 . $f(x, y) = \sin(2xy^4)$; f_{xyt}

27 . $f(x, y) = \text{In}(3x - 2y)$; f_{yxy}

28 . $f(x, y, z) = x^2y + y^2z - 3\sqrt{xz}$; f_{xyz}

29. $f(x, y, z) = \cos(x + 2y + 3z)$; f_{zzx}

30 $f(x, y, z) = e^{xy}\sin z$; f_{zxyx} .

DIFFERENTILITY AND THE GRADIENT

In this section we discuss the notion of the differentiability of a function of several variables. There are several ways to introduce this subject and the way we have chosen is designed to illustrate the great similarities between differentiation of functions of one variable and differentiation of functions of several variables .

We begin with a function of one variables,

Y =f (x).

If f is differentiable , then

$F'(x)=\frac{dy}{dx}=\lim_{\Delta x\to o}\frac{\Delta y}{\Delta x}$

Then if we define the new function $\in(\Delta x)$ by

$$\in(\Delta x)=\frac{\Delta y}{dx}-f'(x),$$

We have

$$\lim_{\Delta x\to o}\in(\Delta x)=\lim_{\Delta x\to o}\left(\frac{\Delta y}{\Delta x}-f'(x)\right)=\lim_{\Delta x\to o}\frac{\Delta y}{\Delta x}-f'(x)$$

$= f'(x) - f'(x) = 0$.

Multiplying both sides of (2) by Δx and rearranging terms, we obtain

$$\Delta y=f'(x)\Delta x+\in(\Delta x)\ \Delta x\,.$$

Note the here Δy depends on both Δx and x. Finally, since $\Delta y=f'(x+\Delta x)-f(x)$, we obtain

$$f'(x+\Delta x)-f(x)=f'(x)\Delta x+\in(\Delta x)\ \Delta x\,.$$

Why did we do all this? We did so in order to be able to state the following alternative definition of differentiability of a function f of one variable.

Definition 1 ALTERNATIVE DEFINITON OF DIFFRENTIABILITY OF A FUNCTION OF ONE VARIABLE Let f be function of one variable . Then f is differentiable at a number x if there is function f' (x) and a function $g(\Delta x)$ such that

$$f'(x+\Delta x)-f(x)=f'(x)\Delta x+g(\Delta x),$$

Where $\lim_{\Delta x\to o[}g(\Delta x)/(\Delta x)]=0$.

We will soon show how the definition (5) can be extended to a function of two or more variables. First, we give a definition

Definition 2 DIFFRENTIABILITY OF A FUNCTION OF TOW VARIABLES: Let f be a real –valued

Function of two variables that is defied in a neighborhood of a point (x, y)and such that $f_x(x, y)$and $f_y(x, y)$ exist.Then f is differentiable a (x, y) if there exist function$\in_1$ (Δx , Δy)and $\in_2$ (Δx , Δy) such that

$$f\ (x+\ \Delta x\ ,y+\ \Delta y\ )-f\ (x\ ,y\ )=\ f_x(x,y\ )\Delta y+\in_1(\Delta x\ ,\Delta y\ )+\in_2(\Delta x\ ,\Delta y\ )\ \Delta y,$$

Where

$$\lim_{(\Delta x,\Delta y)\to(o,0)}\in_1(\Delta x\ ,\Delta y\ )=0\ and\quad \lim_{(\Delta x,\Delta y)\to(o,0)}\in_2(\Delta x\ ,\Delta y\ )=0.$$

DIFFERENTI A BILTY AND THE GRADIENT

In this section we discuss the notion of the differentiability of a function of several variables. There are several ways to introduce this subject and the way we have chosen is designed to illustrate the great similarities between differentiation of functions of several variables.

We being with a function of one variables.

Y = f(x)

If f is differentiable , then

$$f'(x)=\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \qquad (1)$$

Then if we define the new function $\in(\Delta x)$ by

$$\in(\Delta x)=\frac{\Delta y}{\Delta x}-\ f'(x), \qquad (2)$$

We have

$$\lim_{\Delta x\to 0}\in(\Delta x)=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}-f'(x)\right)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}-f'(x)=f'(x)-\ f'(x)=0(3)$$

Multiplying both sides of (2) by Δx and rearranging terms , we obtain

$$\Delta y = f'(x)\Delta x + \in(\Delta x)\,\Delta x$$

Note that hereΔy depends on both Δx and x Finally ,since $\Delta y = f(x+\Delta x) - f(x)$, we obtain

$$f(x+\Delta x) - f(x) = f'(x)\Delta x + +\in(\Delta x)\,\Delta x \qquad (4)$$

Why did we do all this ? We did so in order to be able to state the following alternative definition of differentiability of a function f of one variable

Definition 1 ALTRNATIVE DEFINTION OF DIFFERENTIABILITY OF A FUNCTION OF ONEVARIABLE Let f be a function f of one variable Then f is differentiable at a number x if there is a function f' (x) and a function g (Δx) such that

$$f(x+\Delta x) - f(x) = f'(x)\Delta x + g(\Delta x)(5)$$

Where $\lim_{\Delta x \to 0} \left[\frac{g(\Delta x)}{\Delta x}\right) = 0$

We will soon show how the definition (5) can be extended to two or more variables . First , we give a definition .

Definition 2 DIFFERENTIABILITY OF A FUNCTION OF TOW VARIABLE S Let f be areal –valued function f of two variable s that $f_x(x,y)$and $f_y(x,y)$ exist Then f is differentiable at (x, y) if there exist function f' (x) and a functions$\in_1(\Delta x,\Delta y)$and $\in_2(\Delta x,\Delta y)$such that

$$\begin{aligned} f(x+\Delta x, y+\Delta y) - f(x,y) \\ &= f_x(x,y)\Delta x + f_y(x,y)\Delta y + \in_1(\Delta x,\Delta y)\Delta x \\ &+ \in_2(\Delta x,\Delta y)\Delta y \end{aligned} \qquad (6)$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \in_1(\Delta x,\Delta y) = 0 \quad and \quad \lim_{(\Delta x,\Delta y)\to(0,0)} \in_2(\Delta x,\Delta y) = 0 \qquad (7)$$

EXAMPLE 1 Let f (x, y) = xy . Show that f is differentiable at every point (x , y) in R^2

Solution

$$\begin{aligned} f(x+\Delta x, y+\Delta y) - f(x,y) &= (x+\Delta x)(y+\Delta y) - xy = xy + y\Delta x + x\Delta y + \Delta x\Delta y - xy \\ &= y\Delta x + x\Delta y + \Delta x\Delta y \end{aligned}$$

Now $f_x = y$ and $f_y = x$ so we have

$$f\,(x + \Delta x, y + \Delta y) - f\,(x, y\,) = f_x(x, y)\Delta x + \quad f_y(x, y)\Delta y + +\Delta y\Delta x + 0.\,\Delta y$$

Setting$\in_1 (\Delta x\,, \Delta y) = \Delta y$ and $\in_2 (\Delta x\,, \Delta y) = 0$ we see that

$$\lim_{(\Delta x\,,\Delta y)\to(0\,,0)} \in_1 (\Delta x\,, \Delta y) = \lim_{(\Delta x\,,\Delta y)\to(0\,,0)} \in_2 (\Delta x\,, \Delta y) = 0$$

This result shows that f (x, y) = xy is differentiable at every point in R^2

We now rewrite our definition of differentiability in a more compact from Since a point(x, y) is a vector inR^2 we will write (as we have done before) x =(x , y) .Then if z =f (x , y) we can simply write

Z = f (x) .

Similarly , if w = f (x, y , z)we may write

W= f (x) ,

Where x is the vector (x, y , z). With this notation we may use the symbol Δx to denote the vector$(\Delta x\,, \Delta y)$in R^2 or$(\Delta x\,, \Delta y\,, \Delta z)$ inR^2

Next , we write

$g(\Delta x) = \in_1 (\Delta x\,, \Delta y)\Delta x + \in_2 (\Delta x\,, \Delta y)\,\Delta y (8)$

Note that$(\Delta x\,, \Delta y) \to (\,0\,, 0\,)$ can be written in the compact form $\Delta x \to 0$ .Then if the conditions (7) hold, we see that

$$\downarrow |\Delta x| = \sqrt{\Delta x^2 + \Delta y}$$

$$\lim_{\Delta x\to 0} \frac{|\,g\,(\Delta x)|}{|\Delta x|} \le \lim_{\Delta x\to 0} |\in_1 (\Delta x\,, \Delta y)| \frac{|\Delta x|}{\sqrt{\Delta x^2 + \Delta y^2}} + \lim_{\Delta x\to 0} |\in_2 (\Delta x\,, \Delta y)| \frac{|\Delta x|}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$\frac{|\Delta x|}{\downarrow \sqrt{\Delta x^2 + \Delta y^2}} \le 1$$

$$\le \lim_{\Delta x\to 0} |\in_1 (\Delta x\,, \Delta y)| + \lim_{\Delta x\to 0} |\in_2 (\Delta x\,, \Delta y)| = 0 + 0 = 0$$

Finally , we have the following important definition

Definition 3 THE GRADIENT Let f be a function f of two variables such that f_x, and f_y, exist at a point x =(x, y) Then the gradient of at f at x , denoted ∇f (x), is given by

$$\nabla f(x) = f_x(x\,,y\,)i + f_y(x\,,y\,)\,j \qquad (9)$$

Example 1, f (x, y) =xy $f_x = y$, and $f_y = x$, so that

Using this new notation , we observe that

$$\nabla f\,(x).\,\Delta x = \left(f_x\, i + f_y j\right).\,(\Delta x i + \Delta y\, j\,) = f_x(x\,,y\,)\Delta x + f_y(x\,,y\,)\Delta y$$

Also ,

$$f\,(x + \Delta x, y + \Delta y\,) = f\,\,(x + \,\Delta x).$$

Thus we have the following definition , which is implied Definition 2 .

Definition 4 DIFFERENTIABILITY Let f be a function of two variables that is definition in a neighborhood of a point x = (x, y). Let $\Delta x = (\Delta x, \Delta y)$. If $f_x(x\,,y\,)$ and $f_y(x\,,y\,)$ exist , then f is differentiable at x if there is a function g such that

$$f(x\,+\Delta x\,) - f(x) = \nabla f(x).\,\,\Delta x\,\,+ g(\Delta x), \qquad (10)$$

Where

$$\lim_{(\Delta x \to 0)} \frac{g(\Delta x)}{|\Delta x|} = 0 \qquad (11)$$

Theorem 1: Let f , f_x , and f_y bedefined and con tenuous in a neighborhood of x = (x , y

) Then f is differentiable at x

EXAMPLE 3 Let z =f (x, y)$xy^2 + e^{x^2y^3}$ Show that f is differentiable and calculate ∇f . Find ∇f (1 , 1)

Solution . $\frac{\partial f}{\partial x} = y^2 \cos xy^2 + 2xy^3 e^{x2y^3}$ and $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} = y^2 \cos xy^2 + 2xy^3 e^{x2y^3}$

Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuo us f is differentiable and

$$\Delta f(x,y) = (y^2\cos xy^2 + 2xy^3e^{x2y^3})i + (2xy\ \cos xy^2 + 3x^2y^2e^{x2y^3})\ j$$

At (1 ,1),$\nabla f(1,\ 1) = (\cos 1 + 2e)i + (2\cos 1 + 3e)j$

we showed that the existence of all of its partial derivatives at a point dose not ensure that a function is continuous at that point . However, differentiability (according to Definition 4) does ensure continuity .

Theorem 2 : If f is differentiable at $x_0 = (x_0, y_0)$ then f is continuous at x_0

Proof We must show that $\lim_{\Delta x\to 0} f(x) = f(x_0)$ But if we defineΔx by $\Delta x = x - x_0$ this is the same as showing that

$$\lim_{\Delta x\to 0} f(x_0 + \Delta x) = f(x_0)(12)$$

Since f is differentiable at x_0

$$f(x_0 + \Delta x) - f(x_0) = \nabla f(x_0).\Delta x + g(\Delta x). \qquad (13)$$

But as$\Delta x \to 0$, both terms on the right –hand side of (13) approach zero, so

$$\lim_{\Delta x\to 0}[f(x_0 + \Delta x) - f(x_0)] = 0,$$

Which means that (12) holds and the theorem is proved.

The converse to this theorem is false, as it is in one –variable calculus. That is, there are functions that are continuous, but not differentiable, at a given point . For example, the function

$$f(x,\quad y) = \sqrt[3]{x} + \sqrt[3]{y}$$

Is continuous at any point (x, y) inR^2 But

$$\nabla f(x,y) = \frac{1}{3x^{2/3}}\ i + \frac{1}{3y^{2/3}} j,$$

So f not differentiable at any point (x, y) for which either x or y is zero . That is , is not defined on the x – and y- axes Hence f is not differentiable along these axes .

we showed that

$(f + g)' = f' + g'$ and $(af)' = af'$;

that is , the derivative of the sum of two functions is the sum of the derivatives of the two functions and the derivative of a scalar multiple of a two functions is the scalar times the derivative of the function . These results can be extended to the gradient vector .

Theorem 3 Let f and g be differentiable in a neighborhood of x = (x, y) . Then for every scalar a, ,a.f and f +g are differentiable at x, and

$(i)\nabla(af) = a\nabla f, and$

$(ii)\nabla(f + g) = \nabla f + \nabla g .$

Proof

(i)Form the definition of differentiability (Definition 4), there is a function

$h_1(\Delta x)$such that

$$f\,(x + \Delta x) - f(x) = \nabla f(x).\Delta x + h_1(\Delta x),$$

Where $\lim_{\Delta x\to 0}[(h\_1\ (\Delta x))/|\Delta x|\ ] = 0$ Thus $af(x + \Delta x) - af\,(x) = a\nabla f(x).\Delta x + a\,h_1(\Delta x)$, and

$$\lim_{\Delta x\to 0}\frac{ah_1(\Delta x)}{|\Delta x|} = a\lim_{\Delta x\to 0}\frac{h_1(\Delta x)}{|\Delta x|} = a0 = 0.$$

But

$$a\,\frac{\partial f}{\partial x} = \frac{\partial\,\partial(af)}{\partial x}$$

$$a\nabla f(x) = a\left(f_x i + f_y j\right) = (af)_x i + (af)_y j = \nabla(af)$$

Thus

$af(x + \Delta x) - af(x) = \nabla af(x)\ .\Delta x + \ ah_1(\Delta x),$

Which shows that af is differentiable and$\nabla(af) = a\nabla f$

(ii)As above, there is a function $h_2(\Delta x)$such that $g\,(x, \Delta x) - g(x) = \nabla g(x).\Delta x + h_2(\Delta x)$, where

$$(f + g)(x + \Delta x) - (f + g)(x) = [\ f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]$$

$$= [f(x + \Delta x) - f(x)] + [g(x) + \Delta x) - g(x)]$$

$$= \nabla f(x)\,.\Delta x + h_1(\Delta x) + \nabla g(x).\ \Delta x + h_2(\Delta x)$$

$$= [\nabla f(x) + \nabla g(x)].\Delta x + [h_1(\Delta x) + h_2(\Delta x)],$$

Where

$$\lim_{\Delta x \to 0} \frac{[h_1(\Delta x) + h_2(\Delta x)]}{|\Delta x|} = 0.$$

To complete the proof, we observe that

$$\nabla f(x) + \nabla g(x) = (f_x i + f_y j) + (gi + g_y j) = (g_x + g_y)i + (g_x + g_y)j$$

$$\frac{\partial}{\partial x}(f+g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}$$

$$= (f+g)_x i + (f+g)_y j = \nabla(f+g) .$$

Thus $f + g$ is differentiable and $\nabla(f+g) = \nabla f + \nabla g$.

REMARK. Any function that satisfies conditions (i) and (ii) of Theorem 3 is called a linear mapping or linear operator . Linear operator play an extremely important role in advanced mathematics

All the definitions and theorems in this section hold for functions of three or more variables. We give the equivalent results for functions of three variables below .

Definition 5 THE GRADIENT Let f be scalar function of three variables that f_x, f_y, and f_z exist at a point x = (x, y, z) Then the gradient of f at x, denoted $\nabla f(x)$, is given by the vector

$$\nabla f(x) = f_x(x,y,z)i + f_y(x,y,z)j + f_z(x,y,z)k . \qquad (14)$$

Definition 6 DIFFERENTIABILITY Let f be a function of three variables that is defined in a neighborhood of x = (x, y, z) ,and let $\Delta x = (\Delta x, \Delta y, \Delta z)$. If $f_x(x,y,z) f_y(x,y,z)$, and $f_z(x,y,z)$ exist then f is differentiable at x if there is a function g such that

$$f(x+\Delta x) - f(x) = \nabla f \,.\, \Delta x + g(\Delta x)$$

Where

$$\lim_{|\Delta x| \to 0} \frac{g(\Delta x)}{|\Delta x|} = 0$$

Equivalently , we can write

$$f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z) = f_x(x,y,z)\Delta x + f_y(x,y,z)\Delta y + f_z(x,y,z)\Delta z + g(\Delta x, \Delta y, \Delta z),$$

Where

$$\lim_{(\Delta x,\Delta y,\Delta z)\to(0,0,0)} \frac{g(\Delta x,\Delta y,\Delta z)}{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}} = 0$$

Theorem 1' If f, f_x ,f_y and f_z exist and are continuous in a neighborhood of x = (x, y, z)then f is differentiable at x

Theorem 2' Let f be function of three variables that is differentiable at x_0 Then f is continuous at x_0

EXAMPLE 4 Let f(x, y, z) =xy^2z^3 show that f is differentiable at any point x_0 calculate ∇f , and find $\nabla f(3,-1,2)$

Solution $\frac{\partial f}{\partial x} = y^2z^3, \frac{\partial f}{\partial y} = 2xyz^3$ and $\frac{\partial f}{\partial z} = 3\,xy^2z^2$ Since f , $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are all continuous , we know that f is differentiable and that

$\nabla f = y^2z^3 i + 2xyz^3 j + 3\,xy^2z^2 k$

And

$\nabla f(3,-1,2) = 8i - 48j + 36k$.

Theorem 3' Let f and g be differentiable in a neighborhood of x = (x, y, z)Then for any scalar a , af and f+ g are differentiable at x , and

$(i)\nabla(af) = a\nabla f$, *and*

$(ii)\nabla(f+g) = \nabla f + \nabla g$

We conclude this section with a proof of Theorem 1 . The proof of Theorem 1' is similar

proof of Theorem 1 . We begin by restating the mean value theorem for a function f of one variable .

Mean Value Theorem Let f be continuous on [a, b]anddifferentiableon(a, b)Thenthereisanumbercin(a, b)such that

F (b) - f(a)=f '(c)(b –a).

Now we have assumed that f,f_x, andf_y, are all continuous in a neighborhood N of x = (x, y) ChooseΔx so small that x+Δx N . Then

,y +Δy) − f (x, y)Δf(x) = f(x + Δx

$$\overbrace{[f\ (x+\Delta x, y+\Delta y) - f(x+\Delta x\,, y)] + [f\,(x+\Delta x, y) - f\,(x, y\,)]}^{\text{This lerm was added and subiracted}} \quad (15)$$

If x + is fixed , then f $(x + \Delta x, y)$ is a function of y that
is. Hence by the mean value theorem there is a number c_2 between y and $y + \Delta y$ such that

$$f\,(x + \Delta x, y + \ \Delta y) - f\,(x + \Delta x, y) = f_y(x + \Delta x, c_2)[(y + \Delta y) - y\,]$$
$$= f_y(x + \Delta x, c_2)\Delta y \qquad (16)$$

Similarly, with fixed , f (x, y) is a function of x only , and we obtain

$$f\,(x + \Delta x, y + \ y) - f(x + y) = f_x(c_1\ \ , y)\ \Delta x, (17)$$

Where c_1 is between x and x +Δx Thus using (16) and (17) in (15) , we have

$$\Delta f\,(x) = \ f_x(c_1\ \ , y)\ \Delta x + f_y(x + \Delta x, c_2)\Delta y\,. \qquad (18)$$

Now bothf_x and $f_y$arecontinuousat$x = (x, y)$sosince $c_1$isbetweenxandx + Δx and $c_2$isbetweeny + Δy, weobtain

$$\lim_{\Delta x \to 0} f_x(c_1, y) = f_x(c_1, y) = f_x(x)(19)$$

And

$$\lim_{\Delta x \to 0} f_y(x + \Delta x, c_2) = f_y(c_1, y) = f_y(x)(20)$$

Let

$$\in_1\ (\Delta x) = f_x(c_1, y) - f_x(c_1, y)\,. \qquad (21)$$

From (19) it follows that

$$\lim_{\Delta x \to 0} \in_1\ (\Delta x) = 0 \qquad (22)$$

Similarly , if

$$\in_1\ (\Delta x) = f_y(x + \Delta x, c_2) - f_y(x, y), \qquad (23)$$

Then

$$\lim_{|\Delta x|\to 0} \in_2 (\Delta x) = 0 (24)$$

Now define

$$g(\Delta x) = \in_1 (\Delta x)\,\Delta x + \in_2 (\Delta x)\Delta y \qquad (25)$$

From (22) and (24) it follows that

$$\lim_{|\Delta x|\to 0} \frac{g(\Delta x)}{|\Delta x|} = 0 \ . \qquad (26)$$

Finally , since

$$f_x(c_1, y) - f_x(c_1, y) + \in_1 (\Delta x) \qquad \text{from}(21) \qquad (27)$$

And

$$f_y(x + \Delta x, c_2) = f_y(x, y) + \in_2 (\Delta x), \qquad \text{from}(23) \qquad (28)$$

We may substitute (27) and (28) into (18) to obtain

$$\Delta f(x) = f(x + \Delta x) - f(x) = [f_x(x) + \in_2 (\Delta x)]\Delta x + [f_y(x) + \in_2 (\Delta x)]\Delta y$$
$$= f_x(x)\Delta x + f_y(x)\Delta y + g(\Delta x) = (f_x i + f_y j).\ (\Delta x) + g(\Delta x),$$

Where

$$\lim_{|\Delta x|\to 0} = [g(\Delta x)/|\Delta x|\ ] \to 0, \text{and the proof is (at last)complete .}$$

PROBLEMS

1. Let $f(x, y) = x^2y^2$ Show , by using Definition 2 , that f is differentiable at any point in R^2
2. Let $f(x, y) = x^2y^2$Show , by using Definition 2 , that f is differentiable at any point in R^2
3. Let $f(x, y) =$ be any polynomial in the variables x and y . Showthat f is differentiable

In problems 4- 24 calculate the g gradient of the given function If a point is also given, evaluate the gradient at that point .

4. $f(x, y) = y(x + y)^2$ 5. $f(x, y) = e^{\sqrt{xy}}; (1, 1)$

6 . $f(x, y) = \cos(x - y); \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$ 7. $f(x, y) = \text{In}(2x - y + 1)$

8 . $f(x,y) = \sqrt{x^2 + y^3}$ 9. $f(x,y) = \tan^{-1}\frac{y}{x}$; (3,3)

10. $f(x,y) = y\tan(y - x)$ 11 . $f(x,y) = x^2 \sin hy$

12 . $f(x,y) = \sec(x + 3y)$; (0, 1) 13. $f(x,y) = \frac{x - y}{x + y}$; (3, 1)

14. $f(x,y) = \frac{x^2 - y^2}{x^2 - y^2}$ 15. $f(x,y) = \frac{e^{x2} - e^{-y2}}{3y}$

16. $f(x,y,z) = xyz$; (1,2, 3) 17 . $f(x,y,z) = \sin x \cos y \tan z$; $(\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3})$

18 . $f(x,y,z) = \frac{x^2 - y^2 + z^2}{3xy}$; (1, 2,0) 19 . $f(x,y,z) = x\ln y - z\ln x$

20 . $f(x,y,z) = xy^2 + y^2z^3$; (2, 3, −1) 21. $f(x,y,z) = (y - z)e^{x+2y+3z}$; (−4, −1, 3)

22. $f(x,y,z) = x\sin y\ln z$; (1, 0, 1) 23 . $f(x,y,z) = \frac{x - z}{\sqrt{1 - y^2 + x^2}}$; (0, 0, 1)

24. $f(x,y,z) = x\cosh z - y\sin x$

25. Show that if f and g are differentiablefunction of three variables , then $\nabla(f + g) = \nabla f + \nabla g$

26. Show that if f and g are differentiablefunction of three variables , then fg is differentiable and

$\nabla(fg) = f(\nabla g) + g(\nabla f)$.

∗ 27. Show that $\nabla f = 0$ if and 0nly if f is constant

∗ 28. Show that $\nabla f = \nabla g$, then there is a constant c for which $f(x,y) = g(x,y) + c$ [Hint : Use the result of Problem 27.]

∗ 29 . What is the most general function f such that $\nabla f(x) = x$ for every x in R^2 ?

∗ 30 . Let $f(x,y) = \begin{cases} (x^2 + y^2)\sin\frac{1}{\sqrt{x^2 + y^2}} & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(a) Calculate $f_x(0,0)$ and $f_y(0,0)$

(b)Explain why f_x and f_y are not continuous at $(0, 0)$

(c) Show that f is differentiable at$(0 ,0)$

31. Suppose that f is differentiable function of one variable and g is a differentiable function of three variables. Show that $f^\circ g$ is differentiable and $\nabla f^\circ g = f'(g)\nabla g$

THE CHAIN RULE

In this section we derive the chain rule for function s of two and three variables Let us recall the chain rule for the composition of two f unctions ofone variable :

Let $y = f(u)$ and $u = g(x)$ and assme that f and g are differentiable. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'\big(g(x)\big)g'(x) \quad (1)$$

If z =f (x, y) is a function of two variables , then there are two versions of the chain rule

Theorem 1 CHAIN RULE Let z =f (x, y) be differentiable and suppose that x =x (t) and y= y(t). Assume further that dx/dt and dy/dt exist and are continuous Then z can be written as a function of the parameter t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = f_x\frac{dx}{dt} + \frac{dy}{dt} \quad (2)$$

We can also write this result using our gradient If g(t) =x (t) I +y(t)j, then g'(t)= $(dx/dt)i + (dy/dt)j$, and (2) can be written as

$$\frac{d}{dt}f\big(x(t), y(t)\big) = (f \circ g)'(t) = [f(g(t))]' = \nabla f \cdot g'(t) \quad (3)$$

Theorem 2 CHAIN RULE Let z =f (x, y) be differentiable and suppose that x and y are function of the two variables r and s That is , x= x(r, s) and y =(r, s)

Suppose further that $\partial x/\partial r$, $\partial x/\partial s$, $\partial y/\partial r$ and $\partial y/\partial s$ all exist and are continuous . Then z can be written as a function of r and s, and

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad (4)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad (5)$$

We will leave the proofs of these theorems until the end this section .

EXAMPLE 1: Let $z = f(x, y) = xy^2$. Let $x = \cos t$ and $y = \sin t$ Calculate dz/dt

Solution.

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=y^2(-\sin t)+2xy(\cos t)$$

$=(\sin^2 t)(-\sin t)+2(\cos t\,(\sin t)\,(\cos t)$

$=2\sin t\cos^2 t-\sin^3 t$

We can calculate this result another way . Since $z=xy^2$ we have $z=(\cos t)(\sin^2 t)$ Then

$\frac{dz}{dt}=(\cos t)\,2(\sin t\,(\cos t)\;+\;(\sin^2 t)(-\sin t)$

$=2\sin t\cos^2 t-\sin^3 t$

EXAMPLE 2 Let $z=f(x,y)=\sin xy^2$ Suppose that $x=\frac{r}{s}$ and $y=e^{r-s}$ Calculate $\partial z/\partial r$ and $\partial z/\partial s$

Solution

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=(y^2\cos xy^2)\frac{1}{s}+(2xy\cos xy^2)e^{r-s}$$

$$=\frac{e^{2(r-s)}\cos[(\frac{r}{s})e^{2(r-s)}]}{s}+\frac{2r}{s}\left\{\cos[\frac{r}{s}e^{2(r-s)}]\right\}e^{2(r-s)}$$

And

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=(y^2\cos xy^2)\frac{-r}{s^2}+(2xy\cos xy^2)(-e^{r-s})$$

$$=\frac{-re^{2(r-s)}\cos[(\frac{r}{s})e^{2(r-s)}]}{s^2}+\frac{2r}{s}\left\{\cos[\frac{r}{s}e^{2(r-s)}]\right\}e^{2(r-s)}$$

The chain rules given in Theorem 1 and Theorem 2 can easily be extended to functions of three or more variables .

Theorem 1' Let w= f (x, y , z) be a differentiable function If $x=x(t)$, $y=y(t)$, $z=z(t)$, and if dx/dt, dy/dt, and dz/dt exist and are continuous, then

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{dw}{dz}\frac{dz}{dt}\quad(6)$$

Theorem 2' Let w = f (x, y , z) be a differentiable function and let x= x (r, s), y=y (r, s), and z=z (r, s) Then if all indicated partial derivatives exist and are continuous, we have

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}\;(7)$$

And

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\;(8)$$

Theorem 3' Let w = f (x, y , z) be a differentiable function and let x= x (r, s, t), y=y (r, s), and z=z (r, s, t) Then if all indicated partial derivatives exist and are continuous, we have

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\;(9)$$

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

PROBLEMS

In problems 1-11, use the chain rule to calculate dz/dt Check your answer by first writing z or w as a function of t and then differentiating.

1. $z = xy, x = e^t, y = e^{2t}$

2. $z = x^2 + y^2, x = \cos t, y = \sin t$

3. $z = \frac{y}{x}, x = t^2, y = t^3$

4. $z = e^x \sin y, x = \sqrt{t, y} = \sqrt[3]{t}$

5. $z = \tan^{-1}\frac{y}{x}, x = \cos 3t, y = \sin^{5t}$

6. $z = \sinh(x - 2y), x = 2t^2, y = t^2 + 1$

7. $w = x^2 + y^2, z^2, x = \cos t, y = \sin t, z = t$

8. $w = xy - yz + zx, x \neq e^t, y = e^{2t}, z = e^{3t}$

9. $w = \frac{x + y}{z}, x = t, y = t^2, z = t^3$

10. $w = \sin(x + 2y + 3z), x = \tan t, y = \sec t, z = t^5$

11. $w = \text{In}(2x - 3y + 4z), x = e^t, y = \text{Int}, z = \cosh t$

In problems 12-26, use the chain rule to calculate Check the indicated partial derivatives.

12. $z = xy; x = r + s; y = r - s; \partial z/\partial r$ and $\partial z/\partial s$

13 . $z = x^2 + y^2; x = \dfrac{\cos(r + s); y = \sin(r - s); \quad \partial z}{\partial r}$ and $\dfrac{\partial z}{\partial s}$

14. $z = \dfrac{y}{x}; x = e^r; y = \dfrac{e^s \ \partial z}{\partial r}$ and $\dfrac{\partial z}{\partial s};$

15. $z = \sin\dfrac{y}{x}; x = \dfrac{r}{s}; y = \dfrac{s}{r}; \dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial s}$

16. $z = \dfrac{e^{x+y}}{e^{x-y}}; \ x = \text{In}\, rs; y = \text{In}\ \dfrac{r}{s}; \ \dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial s}$

17. $z = x^2y^3; x = r - s^2; y = 2s + r; \ \partial z/\partial r$ and $\partial z/\partial s$

18 . $w = x + y + z; x = rs; y = r + s; z = r - s; \ \partial w/\partial r$ and $\partial w/\partial s$

19 . $w = \dfrac{xy}{z}; x = r, y = s, z = t; \dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$

20. $w = \dfrac{xy}{z}; x = r + s, y = t - r, z = s - 2t; \dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$

21 . $w = \sin xyz; x = s^2r, y = r^2s, z = r - s; \ \dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$

22. $w = \sinh(x + 2y + 3z); x = \sqrt{r + s, y} = \sqrt[3]{s - t, z} = \dfrac{1}{r + t'}, \dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$

23. $w = xy^2 + yz^2; x = rst, y = \dfrac{rs}{t}, z = \dfrac{1}{rst}; \dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$

24. $w = \text{In}(x + 2y + 3z); x = rt^3 + s; y = t - s^5, z = e^{r+s}; \partial w/\partial r, \partial w/\partial s$ and $\partial w/\partial t$

25. $w = e^{\frac{xy}{z}}; x = r^2 + t^2, y = s^2 - t^2, z = r^2 + s^2; \dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$

$*$ 26 . $u = xy + w^2 - z^3; x = t + r - q, y = q^2 + s^2 - t + r, z = \dfrac{qr + st}{r^2}, w = \dfrac{r - s}{t + q}; \dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial s}, \dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial q}$

18 .7 TANGENT PLANES, NORMAL LINES, AND GRADIENTS

Let z= f(x, y) be a function of two variables . As we have seen, the graph of f is a surface in More general, the graph of the equation F(x , y, z) =0 is a surface in The surface F(x , y, z) =0 is called differentiable at appoint if all exist and are continuous at In a differentiable curve has a unique tangent line at each point . In a differentiable surface in has a unique tangent plane at each point at which are not all zero . We will formally define what we mean by a tangent plane to a surface after a bit, although it should be easy enough to visualize (see Figure 1). We note here that not every surface has a tangent plane at every point . For example, the cone has no tangent plane at the origin (see Figure 2) .

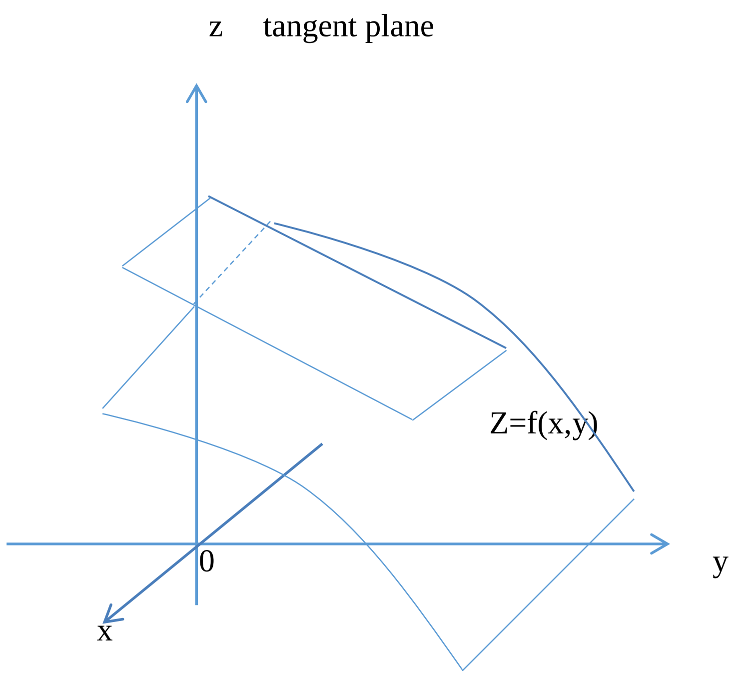


figure 1

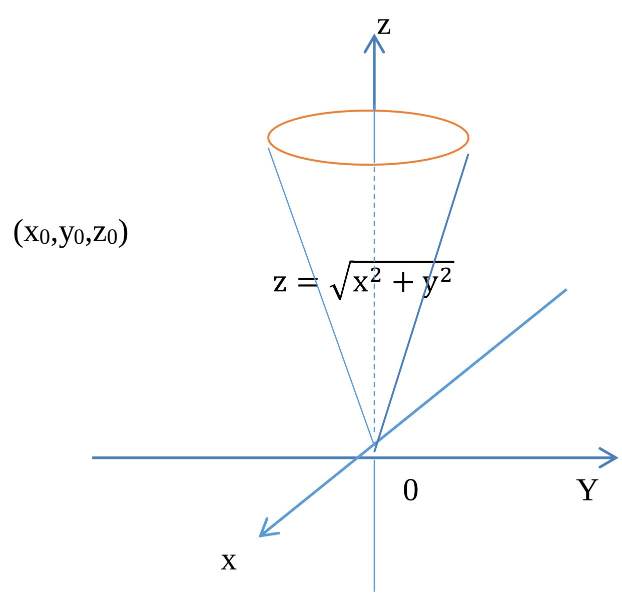


figure 2

Assume that the surface S given by F (x, y, z)=0 is differentiable . Let C be any curve lying on S. That is, C can be given parametrically by g (t)=x(t)I + y(t) j+ z(t) k.(Recall from Definition , the definition of a curve, F(x, y, z) can be written as a function of t, and from of the chain rule [equation (18. 6. 3)] we have

$$F'(t) = \nabla F . g'(t)$$

But since F (x(t), y(t),z (t))=0 for all t since (x(t), y(t),z (t)) is on S we see that F'(t)=0 for all t. But g '(t) is tangent to the curve number t. Thus (1) implies the following :

> The gradient of F at a point $x_0 = (x_0, y_0, z_0)$ on S is orthogonal to tangent vector at x_0 to any curve C remaining on S and passing through x_0

This statement is illustrated in Figure 3

surface

f(x,y,z)=0

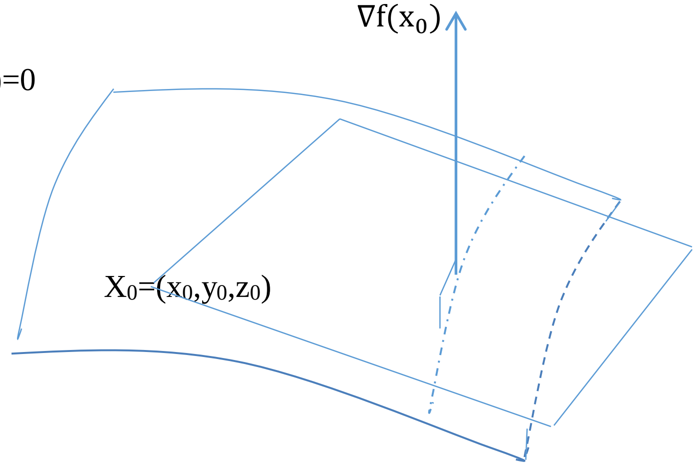


FIGURE 3

Thus is we think of all the vectors tangent to a surface at a point x_0 as constituting a plane, then $\nabla F(x_0)$ is normal vector to that plane . This motivates the following definition.

Definition 1 TANGENT PLANE AND NORMALLINE Let F be differentiable at $x_0 = (x_0, y_0, z_0)$ and let the surface S be defined by F(x , y, z)=0

(i)The tangent plane to S at$(x_0, y_0, z_0)$is the plane passing though the point $(x_0, y_0, z_0)$with normal vector $\nabla F(x_0)$

(ii)The normal line to S at x_0 is the line passing though x_0 having the same direction $\nabla F(x_0)$

EXAMPLE 1 Find the equation of the tangent plane and symmetric equations of the normal line to the ellipsoid $x^2 + \left(\frac{y^2}{4}\right) + \left(\frac{z^2}{9}\right) = 3$ at the point (1, 2, 3).

$x^2 + \left(\frac{y^2}{4}\right) + \left(\frac{z^2}{9}\right) - 3 = 0$ we haveSolution . Since $F(x, y, z) =$

$$\nabla F = \frac{\partial F}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = 2xi + \frac{y}{2} j + \frac{2z}{9} k$$

Then $\nabla F\,(1,2,3) = 2i + j + \frac{2}{3}\,k$, and the equation of the tangent plane is

$$2(x-1) + (y-2) + \frac{2}{3}\,(z-3) = 0,$$

Or

$$2x + y + \frac{2}{3}\,z = 6$$

The normal line is given by

$$\frac{x-1}{2} = y - 2 = \frac{3}{2}\,(z-3)$$

The situation is even simpler if we can write the surface in the form z= f(x , y).

That is , the surface is the graph of function of two variables. Then F(x, y, z)=

$$F_x = f_x\,, \quad F_y = f_y\,, \quad F_z = -1,$$

And the normal vector N to the tangent plane is

$$N = f_x\,(x_0, y_0)i + f_y\,(x_0, y_0)\,j - k \qquad (2)$$

REMARK . One interesting consequence of this fact is that if z = f(x, y)and if $\nabla f(x_0, y_0) = 0$, then the tangent plane to the surface at $(x_0, y_0, f(x_0, y_0)), N = (\partial f/\partial x)i + (\partial f/\partial y)j - k = -k$. is parallel to the xy-plane (i.e.,it is horizontal). This occurs because at Thus the z-axis is normal to the tangent plane.

EXAMPLE 2 Find the tangent plan and normal line to the surface z $=x^3y^5$ at the point (2, 1, 8)

Solution N=$\left(\frac{\partial f}{\partial x}\right) i + \left(\frac{\partial f}{\partial y}\right) j - k = 3x^2y^5 i + 5x^3y^4 j - k = 12i + 40j - k$ Then the tangent plane is given by

$$12(x-2) + 40(y-1) - (z-8) = 0,$$

Or

$$12x + 40y - z = 56$$

Symmetric equations of the normal line are

$$\frac{x-2}{12}=\frac{y-1}{40}=\frac{z-8}{-1}$$

We can write the equation of the tangent plane to a surface = z f(x ,y) so that it looks like the equation of the tangent line to a curve in This will further illustrate the connection between the derivative of a function of one variable and the gradient .Recall from Section 17. 5 that if p is appoint on a plane and N is normal vector, then if Q denotes any other point on the plane, the equation of the plane can be written

$$\overrightarrow{PQ}\,.\quad N = 0\,. \tag{3}$$

In this case, since z = f (x ,y), ap0int on the surface takes the form (x , y , z)=(x , y , f(x, y)). Then since $N = f_x i + f_y j - k$, the equation of the tangent plane at $(x_0, y_0\ f(x_0, y_0))$ becomes , using (3),

$$0 = [(x, y, z) - (x_0, y_0, z_0)]\ .\ (\ f_x + f_{y,} - 1\ )$$

$$= (x - x_0, y - y_0, z - z_0)\ .\ (\ f_x + f_{y,} - 1)$$

$$= (x - x_0)\ f_x + (y - y_0) f_{y,} - (z - z_0) \tag{4}$$

We can rewrite (4) as

$$z = f(x_0, y_0) + (x - x_0)\ f_x + (y - y_0) f_{y,} \tag{5}$$

Denote (x_0, y_0) byx_0 and (x , y) by Then (5) can be written as

$$z = f(x_0) + (x - x_0)\ .\ \nabla f(x_0). \tag{6}$$

Recall the if y = f (x) is differentiable at x_0 then the equation of the tangent line to the curve at the point $(x_{0,}\ f(x_0))$ is given by

$$\frac{y - f(x_0)}{x - x_0} = f'(x_0)\,,$$

Or

$$y = f(x_0) + (x - x_0) f'(x_0). \tag{7}$$

This similarity between (6) and (7) illustrates quite vividly the important of the gradient vector of a function of several variables as the generalization of the derivative of a function of one variable .

PROBLEMS

In Problems 1-16 find the equation of the tangent plane and symmetric equations of the normal line to given surface at the given point .

1. $x^2 + y^2 + z^2 = 1; (1, 0, 0)$

2. $x^2 + y^2 + z^2 = 1; (0, 1, 0)$

3. $x^2 + y^2 + z^2 = 1; (0, 0, 1)$

4. $x^2 + y^2 + z^2 = 1; (1, 1, 1)$

5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3 ; (a , b, c)$

6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3 ; (-a , b, -c)$

7. $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = 6; (4, 1, 9)$

8. $ax + by + cz = d; \left(\frac{1}{a} , \frac{1}{b}, \frac{d-2}{c}\right)$

9. $xyz = 4; (1, 2, 2)$

10. $xy^2 + yz^2 + zx^2 = 1; (1, 1, 1)$

11. $4x^2 + y^2 + 5z^2 = 15; (3, 1, -2)$

12. $xe^y - ye^3 = 1; (1, 0, 0)$

13. $\sin xy - 2 \cos yz = 0; \left(\frac{\pi}{2} , 1, \frac{\pi}{3}\right)$

14 . $x^2 + y^2 + 4x + 2y + 8z = 7; (2, -3, -1)$

15. $e^{xyz} = 5; (1, 1, \text{In } 5)$

16. $\sqrt{\frac{x + y}{z - 1}} = 1; (1, 1, 3)$

In Problems 7-24, write the equation of the tangent plane in the form (6) and find the symmetric equations of the normal line to given surface

17 . $z = xy^2; (1, 1, 1)$

18. $z = \text{In}(x - 2y); (3, 1, 0)$

19. $z = \sin(2x + 5y); \left(\frac{\pi}{8}, \frac{\pi}{20}, 1\right)$

20 . $z = \sqrt{\frac{x + y}{x - y}}; (5, 4, 3)$

21. $z = \tan^{-1}\frac{y}{x} ; \left(-2, 2 , -\frac{\pi}{4}\right)$

22. $z = \sin hxy^2; (0, 3, 0)$

23. $z = \sec(x - y); \left(\frac{\pi}{2}, \frac{\pi}{6}, 2\right)$

24. $z = e^x \cos y + e^y \cos x ; \left(\frac{\pi}{2}, 0 , e^{\frac{\pi}{2}}\right)$

$*$ 25. Find the two points of intersection of the surface $z = x^2 + y^2$ and the line

$$\frac{x - 3}{1} = \frac{y + 1}{-1} = \frac{z + 2}{-2} .$$

DIRECTIONAL DERIVES AND THE GRADIENT

Let us take another look at the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function z =(x, y) . We have

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \tag{1}$$

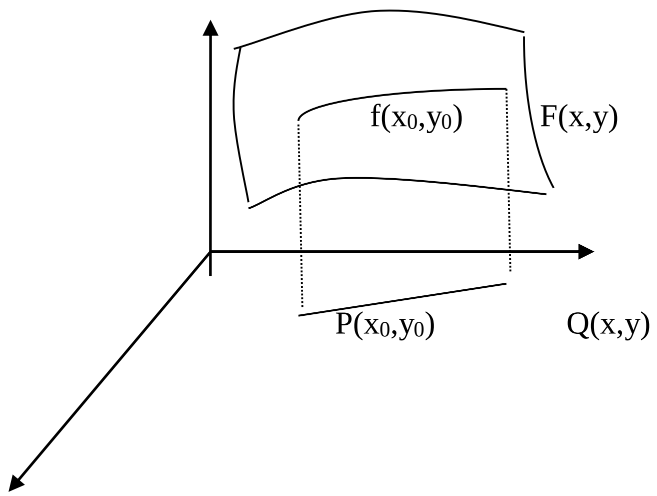
This measures the rate of f as we approach the point (x_0, y_0) a long a vector parallel to the x-axis [since $(x_0 + \Delta x, y_0) - (x_0, y_0) = (\Delta x, 0) = \Delta x i$] Similarly

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \tag{2}$$

measures the rate of change of f as we approach the point (x_0, y_0) a long a vector parallel to the y – axis .

It is frequently of interest to compute the rate of change of f as we approach (x_0, y_0) along a vector that is not parallel to one of the coordinate axes The situation is depicted in Figure 1. Suppose that (x, y) approaches the fixed point (x_0, y_0) along the line segment joining them, and let t denote the distance between the two points. We want to determine the relative rate of change in f with respect to a change in f with respect to a change in t . Let u denote a unit vector with the initial point at (x_0, y_0) and parallel to $\overrightarrow{PQ}$ (see Figure 2) . Since u and $\overrightarrow{PQ}$ are parallel , there is , by Theorem a value of t such that

$$\overrightarrow{PQ} = tu \tag{3}$$



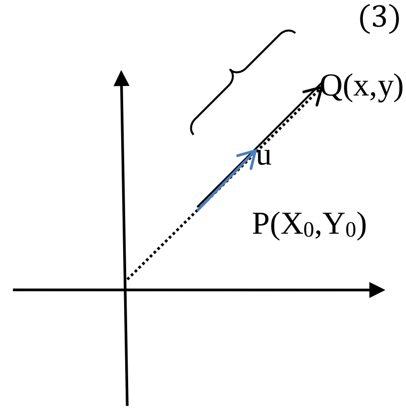


Figure 2

Figure 1

Not that $t > 0$ if u and $\overrightarrow{PQ}$ have the same direction and $t < 0$ if u and $\overrightarrow{PQ}$ have opposite directions . Now

$$\overrightarrow{PQ} = (x - x_0)i + (y - y_0)j \ , \qquad (4)$$

And since u is a unit vector, we have

$$u = \cos\theta i + \sin\theta j \qquad (5)$$

Where θ is the direction of u Thus inserting (4) and (5) into (3), we have

$$(x - x_0)i + (y - y_0)j = t\cos\theta i + t\sin\theta j,$$

or

$$x = x_0 + t\cos\theta$$

$$y = y_0 + t\sin\theta \qquad (6)$$

The equations (6) are the parametric equations of the line passing through p and Q Using (6) , we have

$$z = f(x, y) = f(x_0 + t\cos\theta , y_0 + t\sin\theta) \qquad (7)$$

Remember that is fixed —it is the direction of approach Thus$(x, y) \to (x_0 , y_0)$ along $\overrightarrow{PQ}$ is equivalent to $t \to 0$ in (7) Hence to compute the instantaneous rate of change of f as $(x, y) \to (x_0 , y_0)$ along the vector $\overrightarrow{PQ}$ we need compute But by the chain rule ,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}(x, y)\frac{dx}{dt} + \frac{\partial f}{\partial y}(x, y)\frac{dy}{dt}$$

Or

$$\frac{dz}{dt} = f_x(x , y)\cos\theta + f_y(x , y)\sin\theta \qquad (8)$$

And

$$\frac{dz}{dt} = [f_x(x_0 + t\cos\theta , y_0 + t\sin\theta]\cos\theta$$
$$+ [f_y(x_0 + t\cos\theta , y_0 + t\sin\theta]\sin\theta \qquad (9)$$

If we set we obtain the instantaneous rate of change of f in the direction $\overrightarrow{PQ}$ at the point (x_0, y_0). That is ,

$$\frac{dz}{dt}|_{t=0} = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta. \qquad (10)$$

But (10) can be written [using (5)] as

$$\frac{dz}{dt}|_{t=0} = \nabla f(x_0, y_0) . u \qquad (11)$$

This leads to the following definition.

Definition 1 DIRCTIONAL DERIVATIVE Let f be differentiable at a point $x_0 = (x_0, y_0)$ in R^2 and let u be a unit vector . Then the directional derivative of f in the direction u , denoted $f'_u(x_0)$, is given by

$$f'_u(x_0) = \nabla f\,(x_0) .\ u \qquad (12)$$

REMARK 1. Note that if u= I, then $\nabla f\ . u = \partial f/\partial x$ and (12) reduces to the partial derivative $\partial f/\partial x$ Similarly, if u = j, then (12) reduces to $\partial f/\partial y$

REMARK 2. Definition 1 makes sense if f is a function of three variables . Then, of course, u is a unit vector inR^3

REMARK 3. There is another definition of the directional derivative . It is given by

$$f'_u(x_0) = \lim_{h\to 0}\frac{f(x_0 + h_u) - f(x_0)}{h} \qquad (13)$$

It can be shown that if limit in (13) exists , it is equal to$\nabla f\,(x_0)$. u if f is differentiable .

EXAMPL1: Let $z = f\,(x,\ y) = xy^2$ Calculate the directional derivative of f in the direction of the vector v=2i +3j at the point (4, -1)

Solution A unit vector in the direction v is u= $(2/\sqrt{13})i + 3/\sqrt{13})j$ Also, $\nabla f = y^2 i + 2xyj$. Thus

$$f'_u(x, y) = \nabla f(x).u = \frac{2y^2}{\sqrt{13}} + \frac{6xy}{\sqrt{13}} = \frac{2y^2+6xy}{\sqrt{13}}$$

At

$(4,-1), f'_u(4,-1) = -22\sqrt{13}$

EXAMPLE 2 Let $z = f(x, y, z) = x\ln y - e^{xz3}$ Calculate the directional derivative of f in the direction of the vector $v = i - j + 3k$ Evaluate this derivative at the point $(-5,1,-2)$

Solution A unit vector in the direction v is $u = \left(\frac{1}{\sqrt{11}}\right) i - 1/\sqrt{11})j + (3/\sqrt{11})k$, and

$$\nabla f = (\ln y - z^3 e^{xz3})i + \frac{x}{y}j - 3xz^2e^{xz3}\, k$$

Thus

$$f'_u(x) = \nabla f(x) . u = \frac{\ln y - z^3e^{xz3} - \left(\frac{x}{y}\right) - 9xz^2e^{xz3}}{\sqrt{11}},$$

And at $(-5,1,-2)$

$$f'_u(-5,1,-2) = \frac{5+188e^{40}}{\sqrt{11}}$$

PROBLEMS

In problems 1-15 , calculate the directional derivative of the given function at the given point in the direction of the given vector v .

1. $f(x,y) = xy$ at $(2,3)$; $v = i + 3j$

2. $f(x,y) = 2x^2 - 3y^2$ at $(1,-1)$; $v = -i + 2j$

3. $f(x,y) = \ln(x + 3y)$ at $(2,4)$; $v = i + j$

4. $f(x,y) = ax^2 + by^2$ at (c,d); $v = ai + Bj$

5. $f(x,y) = \tan^{-1}\frac{y}{x}$ at $(2,2)$; $v = 3i - 2j$

6. $f(x,y) = \frac{x-y}{x+y}$ at $(4,3)$; $v = -i - 2j$

7. $f(x,y) = xe^y + ye^x$ at $(1,2)$; $v = i + j$

8. $f(x,y) = \sin(2x + 3y)$ at $\left(\frac{\pi}{12}, \frac{\pi}{9}\right)$; $v = -2j + 3j$

9. $f(x,y,z) = xy + yz + xz$ at $(1,1,1)$; $v = i + j + k$

10. $f(x,y,z) = xy^3z^5$ at $(-3,-1,2)$; $v = -i - 2j + k$

11. $f(x, y, z) = \ln(x + 2y + 3z)$ at $(1, 2, 0); v = 2i + j - k$

12. $f(x, y, z) = xe^{yz}$ at $(2, 0, -4); v = -i + 2j + 5k$

13. $f(x, y, z) = x^2y^3 + z\sqrt{x}$ at $(1, -2, 3); v = 5j + k$

14. $f(x, y, z) = e^{-(x^2+y^2+z^2)}$ at $(1, 1, 1); v = i + 3j - 5k$

15. $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ at $(-1, 2, 3); v = i - j + k$

THE TOTAL DIFFERENTAL AND APPROXIMATION

In Section 3.8 we used the notions of increments and differentials to approximate a function .We used the fact that if Δx was small, then

$$f(x + \Delta x) - f(x) = \Delta y \approx f'(x)\Delta x . \qquad (1)$$

We also defined the differential d y b y

$$dy = f'(x)dx = f'(x)\Delta x \qquad (2)$$

(since dx defined to be equal to Δx) . Not that in (2) it is not required that Δx be small

We now extend these ideas to functions of two or three variables.

Definition 1 INCREMNT AND TOTAL DIFFERENTIAL Let $f = f(x)$ be a function of two or three variables, and let $\Delta x = (\Delta x, \Delta y)$ or $(\Delta x, \Delta y, \Delta z)$

(i)The increment of f , denoted Δf , is defined by

$$\Delta f = f(x + \Delta x) - f(x) \qquad (3)$$

(ii)The total differential of f , denoted df, is given by

$$df = \nabla f(x). \Delta x . \qquad (4)$$

Note that equation (4) is very similar in form to equation (2).

REMARK 1. If f is a function of two variables , then (3) and (4) become

$$\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y), \qquad (5)$$

And the total differential is

$$df = f_x(x + \Delta x, y + \Delta y) - f(x, y) \qquad (6)$$

REMARK 1. If f is a function of three variables , then (3) and (4) become

$$\Delta f = f(x + \Delta x, y + \Delta y , z + \Delta z) - f(x, y, z) \quad (7)$$

And

$$df = f_x(x, y, z)\Delta x + f_y(x, y, z)\Delta y + f_z(x, y, z)\Delta z. \quad (8)$$

REMARK 3. Note that in the definition of the total differential , it is not required that be $|\Delta x|$ small .

From Theorems .1 and . 1' and the definition of differentiability , we see that $|\Delta x|$ if is small and if f is differentiable , then

$$\Delta f \approx df. \quad (9)$$

We can use the relation (9) to approximate functions of several variables in much the same way that we used the relation (1) to approximate the values of functions of one variable .

EXAMPLE 1 Use the total differential to estimate $\sqrt{(2.98)^2 + (4.\ 03)^2}$

Solution Let $f(x, y) = \sqrt{x^2 + y^2}$ Then we are asked to calculate f (2.98, 4. 03) We know that$f(3, 4) = \sqrt{3^2 + 4^2} = 5$. Thus we need to calculate Now at (3, 4),

$$\nabla f(x) = \frac{x}{\sqrt{x^2 + y^2}}\, i + \frac{y}{\sqrt{x^2 + y^2}}\, j = \frac{3}{5} i + \frac{4}{5} j .$$

Then using (6), we have

$$df = \frac{3}{5}\Delta x + \frac{4}{5}\Delta y = (0.6)(-0.02) + (0.8)(0.03) = 0.012 .$$

Hence

$$f(3 - 0.02, 4 + 0.03) - f(3,4) = \Delta f \approx df = 0.012,$$

So

$$f(2.98,4.03) \approx f(3,4) + 0.012 = 5.012 .$$

The exact value of $\sqrt{(2.98)^2} + (4.03)^2$ is $\sqrt{8.8804 + 16.2409}$ =$\sqrt{25.1213} \approx 5.012115$, so that $\Delta f \approx 0.012115$ and our approximation is very good indeed .

EXAMPLE 2 The radius of a cone is measured to be 15 cm and the height of the cone is measured to be 25 cm . There is a maximum error o f ± 0.02 cm in the measurement of the radius and ± 0.05 cm in the measurement of the height (a) What is the approximate volume of the cone? (b)what is the maximum error in the calculation of the volume ?

Solution (a) V= $\frac{1}{3}\pi r^2 h \approx \frac{1}{3}\pi(15)^2\, 25 = 1875\pi cm^3 \approx 5890.5cm^3$

(b)$\nabla v = v_r i + v_h\, j = \frac{2}{3}\pi rhi + \frac{1}{3}\pi r^2\, j = \pi(250i + 75j)$Then choosing$\Delta x = 0.02$ and$\Delta y = 0.05$ to find the maximum error , we have

$\Delta v \approx dv = \nabla v.\ \Delta x = \pi[250(0.02) + 75(0.05)] = \pi(5 + 3.75) = 8.75\pi \approx 27.5cm^3$

Thus the maximum error in the calculation is , approximately, $27.5cm^3$, which means that

5890.5 -27.5 <V<5890.5 +27.5,

Or

$5863cm^3 < v < 5918cm^3$

Note that an error of $27.5cm^3$ is only a relative error of 27.5/5890.5 ≈ 0.0047, which is a very small relative error (see p . 158 for discussion of relative error).

EXAMPLE 3. A cylindrical tin can has an inside radius of 5 cm and height of 12cm . The thickness of the is tin is 0.2 cm. Estimate the amount of tin needed to construct the can (in clueing its ends).

Solution .We need to estimate the difference between the ''outer'' and ''inner'' volumes of the can We have V= $\pi r^2 h$The inner volume is $(5^2)(12) = 300\pi cm^3$, and the outer volume is$\pi\,(5.2)^2(12.4)$.The difference is

$\Delta v = \pi\,(5.2)^2(12.4) - 300\pi \approx dv.$

Since $\Delta v = 2\pi rhi + \pi r^2 j = \pi(120 + 25j)$, we have

$dv = \pi\big(120(0.2) + 25(0.4)\big) = 34\pi$.

Thus the amount of tin needed is, approximately, $34\pi cm^3 \approx 106.8cm^3$

PROBLEMS

In problems 1-12, calculate the total differential d f

1. $f(x, y) = xy^3$

2. $f(x, y) = \tan^{-1}\frac{y}{x}$

3. $f(x, y) = \sqrt{\frac{x-y}{x+y}}$

4. $f(x, y) = xe^y$

5. $f(x, y) = In(2x + 3y)$

6. $f(x, y) = \sin(x - 4y)$

7. $f(x, y, z) = xy^2z^s$

8. $f(x, y, z) = \frac{xy}{z}$

9. $f(x, y, z) = In(x + 2y + 3z)$

10. $(x, y, z) = secxy - \tan z$

11. $f(x, y, y) = \cosh(xy - z)$

12. $f(x, y, z) = \frac{x - z}{y + 3x}$

13. Let $f(x, y) = xy^2$

(a)Calculate explicitly he difference $\Delta f - df$

(b)Verify your answer by calculating $\Delta f - df$ at the point $(1, 2)$, where $\Delta x = -0.01$ and $\Delta y = 0.03$.

$*$ 14. Repeat the steps of problem 13 for the function $f(x, y) = x^3y^2$.

In problems 15 -23, use the total differential to estimate the given number .

15. $\frac{3.01}{5.99}$

16. $19.8\sqrt{65}$

17. $\sqrt{35.6}\sqrt[3]{64.08}$

18. $(2.01)^4\ (3.04)^7\ -(2.01)\ (3.04)^9$

19. $\sqrt{\frac{5.02 - 3.96}{5.02 + 3.96}}$

20. $((4.95)^2\ + (7.02))^{\frac{1}{5}}$

21. $\frac{(3.02)(1.97)}{\sqrt{8.95}}$

22. $\sin(\frac{11\pi}{24})\ \cos\left(\frac{13\pi}{36}\right)$