DIRECTIONAL DERIVES AND THE GRADIENT

Let us take another look at the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function z =(x, y) . We have

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \tag{1}$$

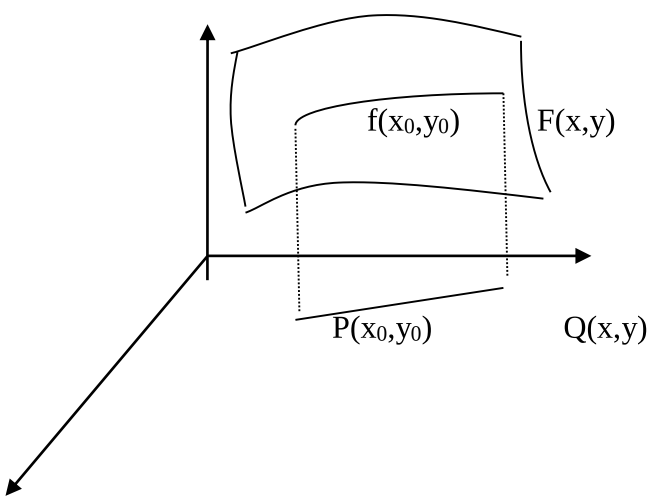
This measures the rate of f as we approach the point (x_0, y_0) a long a vector parallel to the x-axis [since $(x_0 + \Delta x, y_0) - (x_0, y_0) = (\Delta x, 0) = \Delta x i$] Similarly

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \tag{2}$$

measures the rate of change of f as we approach the point (x_0, y_0) a long a vector parallel to the y – axis .

It is frequently of interest to compute the rate of change of f as we approach (x_0, y_0) along a vector that is not parallel to one of the coordinate axes The situation is depicted in Figure 1. Suppose that (x, y) approaches the fixed point (x_0, y_0) along the line segment joining them, and let t denote the distance between the two points. We want to determine the relative rate of change in f with respect to a change in f with respect to a change in t . Let u denote a unit vector with the initial point at (x_0, y_0) and parallel to $\overrightarrow{PQ}$ (see Figure 2) . Since u and $\overrightarrow{PQ}$ are parallel , there is , by Theorem a value of t such that

$$\overrightarrow{PQ} = tu \tag{3}$$



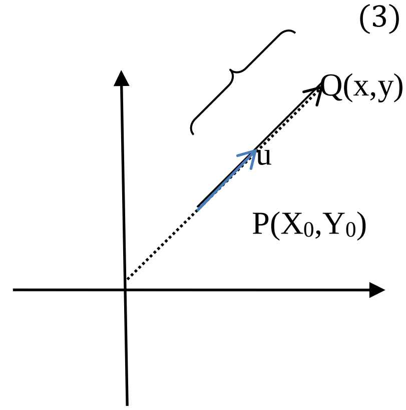


Figure 2

Figure 1

Not that $t > 0$ if u and $\overrightarrow{PQ}$ have the same direction and $t < 0$ if u and $\overrightarrow{PQ}$ have opposite directions . Now

$$\overrightarrow{PQ} = (x - x_0)i + (y - y_0)j \ , \qquad (4)$$

And since u is a unit vector, we have

$$u = \cos\theta i + \sin\theta j \qquad (5)$$

Where θ is the direction of u Thus inserting (4) and (5) into (3), we have

$$(x - x_0)i + (y - y_0)j = t\cos\theta i + t\sin\theta j,$$

or

$$x = x_0 + t\cos\theta$$

$$y = y_0 + t\sin\theta \qquad (6)$$

The equations (6) are the parametric equations of the line passing through p and Q Using (6) , we have

$$z = f(x, y) = f(x_0 + t\cos\theta, y_0 + t\sin\theta) \qquad (7)$$

Remember that is fixed —it is the direction of approach Thus$(x, y) \to (x_0, y_0)$ along $\overrightarrow{PQ}$ is equivalent to $t \to 0$ in (7) Hence to compute the instantaneous rate of change of f as $(x, y) \to (x_0, y_0)$ along the vector $\overrightarrow{PQ}$ we need compute But by the chain rule ,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}(x, y)\frac{dx}{dt} + \frac{\partial f}{\partial y}(x, y)\frac{dy}{dt}$$

Or

$$\frac{dz}{dt} = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \qquad (8)$$

And

$$\frac{dz}{dt} = [f_x(x_0 + t\cos\theta, y_0 + t\sin\theta]\cos\theta$$
$$+ [f_y(x_0 + t\cos\theta, y_0 + t\sin\theta]\sin\theta \qquad (9)$$

If we set we obtain the instantaneous rate of change of f in the direction $\overrightarrow{PQ}$ at the point (x_0, y_0). That is ,

$$\frac{dz}{dt}|_{t=0} = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta. \quad (10)$$

But (10) can be written [using (5)] as

$$\frac{dz}{dt}|_{t=0} = \nabla f(x_0, y_0).u \quad (11)$$

This leads to the following definition.

Definition 1 DIRCTIONAL DERIVATIVE Let f be differentiable at a point $x_0 = (x_0, y_0)$ in R^2 and let u be a unit vector . Then the directional derivative of f in the direction u , denoted $f'_u(x_0)$, is given by

$$f'_u(x_0) = \nabla f(x_0).\, u \quad (12)$$

REMARK 1. Note that if u= I, then $\nabla f\ .u = \partial f/\partial x$ and (12) reduces to the partial derivative $\partial f/\partial x$ Similarly, if u = j, then (12) reduces to $\partial f/\partial y$

REMARK 2. Definition 1 makes sense if f is a function of three variables . Then, of course, u is a unit vector inR^3

REMARK 3. There is another definition of the directional derivative . It is given by

$$f'_u(x_0) = \lim_{h\to 0}\frac{f(x_0 + h_u) - f(x_0)}{h} \quad (13)$$

It can be shown that if limit in (13) exists , it is equal to$\nabla f(x_0).\ u$ if f is differentiable .

EXAMPL1: Let $z = f(x, y) = xy^2$ Calculate the directional derivative of f in the direction of the vector v=2i +3j at the point (4, -1)

Solution A unit vector in the direction v is u= $(2/\sqrt{13})i + 3/\sqrt{13})j$ Also, $\nabla f = y^2 i + 2xyj$. Thus

$$f'_u(x, y) = \nabla f(x).u = \frac{2y^2}{\sqrt{13}} + \frac{6xy}{\sqrt{13}} = \frac{2y^2+6xy}{\sqrt{13}}$$

At

$(4,-1), f'_u(4,-1) = -22\sqrt{13}$

EXAMPLE 2 Let $z = f(x, y, z) = x\text{In}\, y - e^{xz3}$ Calculate the directional derivative of f in the direction of the vector $v = i - j + 3k$ Evaluate this derivative at the point $(-5,1,-2)$

Solution A unit vector in the direction v is $u = \left(\frac{1}{\sqrt{11}}\right) i - 1/\sqrt{11})j + (3/\sqrt{11})k$, and

$$\nabla f = (\text{In}\, y - z^3 e^{xz3})i + \frac{x}{y} j - 3xz^2 e^{xz3}\, k$$

Thus

$$f'_u(x) = \nabla f(x) \,.\, u = \frac{\text{In}\, y - z^3 e^{xz3} - \left(\frac{x}{y}\right) - 9xz^2 e^{xz3}}{\sqrt{11}},$$

And at $(-5,1,-2)$

$$f'_u(-5,1,-2) = \frac{5+188e^{40}}{\sqrt{11}}$$

PROBLEMS

In problems 1-15 , calculate the directional derivative of the given function at the given point in the direction of the given vector v .

1. $f(x,y) = xy$ at$(2,3)$; $v = i + 3j$

2. $f(x,y) = 2x^2 - 3y^2$ at $(1,-1)$; $v = -i + 2j$

3. $f(x,y) = \text{In}(x + 3y)at(2,4)$; $v = i + j$

4. $f(x,y) = ax^2 + by^2$ at (c,d); $v = ai + Bj$

5. $f(x,y) = \tan^{-1}\frac{y}{x}$ at $(2,2)$; $v = 3i - 2j$

6. $f(x,y) = \frac{x-y}{x+y}$at $(4,3)$; $v = -i - 2j$

7. $f(x,y) = xe^y + ye^x$ at$(1,2)$; $v = i + j$

8. $f(x,y) = \sin(2x + 3y)$ at $\left(\frac{\pi}{12}, \frac{\pi}{9}\right)$; $v = -2j + 3j$

9. $f(x,y,z) = xy + yz + xz$ at $(1,1,1)$; $v = i + j + k$

10. $f(x,y,z) = xy^3z^5$ at$(-3,-1,2)$; $v = -i - 2j + k$

11. $f(x,y,z) = \text{In}\ (x+2y+3z)$ at $(1,2,0)$; $v = 2i + j - k$

12. $f(x,y,z) = xe^{yz}$ at $(2,0,-4)$; $v = -i + 2j + 5k$

13. $f(x,y,z) = x^2y^3 + z\sqrt{x}$ at $(1,-2,3)$; $v = 5j + k$

14. $f(x,y,z) = e^{-(x^2+y^2+z^2)}$ at $(1,1,1)$; $v = i + 3j - 5k$

15. $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ at $(-1,2,3)$; $v = i - j + k$

THE TOTAL DIFFERENTAL AND APPROXIMATION

In Section 3.8 we used the notions of increments and differentials to approximate a function .We used the fact that if Δx was small, then

$$f(x+\Delta x) - f(x) = \Delta y \approx f'(x)\Delta x\,. \qquad (1)$$

We also defined the differential d y b y

$$dy = f'(x)dx = f'(x)\Delta x \qquad (2)$$

(since dx defined to be equal to Δx) . Not that in (2) it is not required that Δx be small

We now extend these ideas to functions of two or three variables.

Definition 1 INCREMNT AND TOTAL DIFFERENTIAL Let f = f(x) be a function of two or three variables, and let Δx =$(\Delta x, \Delta y)$or $(\Delta x, \Delta y, \Delta z)$

(i)The increment of f , denoted Δf , is defined by

$$\Delta f = f(x+\Delta x) - f(x) \qquad (3)$$

(ii)The total differential of f , denoted df, is given by

$$df = \nabla f(x).\Delta x\,. \qquad (4)$$

Note that equation (4) is very similar in form to equation (2).

REMARK 1. If f is a function of two variables , then (3) and (4) become

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y), \qquad (5)$$

And the total differential is

$$df = f_x(x+\Delta x, y+\Delta y) - f\,(x,y) \qquad (6)$$

REMARK 1. If f is a function of three variables , then (3) and (4) become

$$\Delta f = f(x + \Delta x, y + \Delta y , z + \Delta z) - f(x, y, z) \quad (7)$$

And

$$df = f_x(x, y, z)\Delta x + f_y(x, y, z)\Delta y + f_z(x, y, z)\Delta z. \quad (8)$$

REMARK 3. Note that in the definition of the total differential , it is not required that be $|\Delta x|$ small .

From Theorems .1 and . 1' and the definition of differentiability , we see that $|\Delta x|$ if is small and if f is differentiable , then

$$\Delta f \approx df. \quad (9)$$

We can use the relation (9) to approximate functions of several variables in much the same way that we used the relation (1) to approximate the values of functions of one variable .

EXAMPLE 1 Use the total differential to estimate $\sqrt{(2.98)^2 + (4.\ 03)^2}$

Solution Let $f(x, y) = \sqrt{x^2 + y^2}$ Then we are asked to calculate f (2.98, 4. 03) We know that$f(3, 4) = \sqrt{3^2 + 4^2} = 5$. Thus we need to calculate Now at (3, 4),

$$\nabla f(x) = \frac{x}{\sqrt{x^2 + y^2}}\, i + \frac{y}{\sqrt{x^2 + y^2}}\, j = \frac{3}{5} i + \frac{4}{5} j\,.$$

Then using (6), we have

$$df = \frac{3}{5}\Delta x + \frac{4}{5}\,\Delta y = (0.6)(-0.02) + (0.8)(0.03) = 0.012\,.$$

Hence

$$f(3 - 0.02, 4 + 0.03) - f(3,4) = \Delta f \approx df = 0.012,$$

So

$$f(2.98,4.03) \approx f(3,4) + 0.012 = 5.012\,.$$

The exact value of $\sqrt{(2.98)^2} + (4.03)^2$ is $\sqrt{8.8804 + 16.2409}$ =$\sqrt{25.1213} \approx 5.012115$, so that $\Delta f \approx 0.012115$ and our approximation is very good indeed .

EXAMPLE 2 The radius of a cone is measured to be 15 cm and the height of the cone is measured to be 25 cm . There is a maximum error o f $\pm\, 0.02$ cm in the measurement of the radius and $\pm\, 0.05$ cm in the measurement of the height (a) What is the approximate volume of the cone? (b)what is the maximum error in the calculation of the volume ?

Solution (a) V= $\frac{1}{3}\pi r^2 h \approx \frac{1}{3}\pi(15)^2\, 25 = 1875\pi \mathrm{cm}^3 \approx 5890.5\mathrm{cm}^3$

(b)$\nabla v = v_r i + v_h\, j = \frac{2}{3}\pi rhi + \frac{1}{3}\pi r^2\, j = \pi(250i + 75j)$ Then choosing $\Delta x = 0.02$ and $\Delta y = 0.05$ to find the maximum error, we have

$\Delta v \approx dv = \nabla v.\ \Delta x = \pi[250(0.02) + 75(0.05)] = \pi(5 + 3.75) = 8.75\pi \approx 27.5\mathrm{cm}^3$

Thus the maximum error in the calculation is , approximately, $27.5\mathrm{cm}^3$, which means that

5890.5 -27.5 <V<5890.5 +27.5,

Or

$5863\mathrm{cm}^3 < v < 5918\mathrm{cm}^3$

Note that an error of $27.5\mathrm{cm}^3$ is only a relative error of 27.5/5890.5 ≈ 0.0047, which is a very small relative error (see p . 158 for discussion of relative error).

EXAMPLE 3. A cylindrical tin can has an inside radius of 5 cm and height of 12cm . The thickness of the is tin is 0.2 cm. Estimate the amount of tin needed to construct the can (in clueing its ends).

Solution .We need to estimate the difference between the ''outer'' and ''inner'' volumes of the can We have V= $\pi r^2 h$The inner volume is $(5^2)(12) = 300\pi \mathrm{cm}^3$, and the outer volume is$\pi\ (5.2)^2(12.4)$.The difference is

$\Delta v = \pi\ (5.2)^2(12.4) - 300\pi \approx dv.$

Since $\Delta v = 2\pi rhi + \pi r^2 j = \pi(120 + 25j)$, we have

$dv = \pi\big(120(0.2) + 25(0.4)\big) = 34\pi\ .$

Thus the amount of tin needed is, approximately, $34\pi \mathrm{cm}^3 \approx 106.8\mathrm{cm}^3$

PROBLEMS

In problems 1-12, calculate the total differential d f

1. $f(x, y) = xy^3$

2. $f(x, y) = \tan^{-1}\frac{y}{x}$

3. $f(x, y) = \sqrt{\frac{x-y}{x+y}}$

4. $f(x, y) = xe^y$

5. $f(x, y) = In(2x + 3y)$

6. $f(x, y) = \sin(x - 4y)$

7. $f(x, y, z) = xy^2z^s$

8. $f(x, y, z) = \frac{xy}{z}$

9. $f(x, y, z) = In(x + 2y + 3z)$

10. $(x, y, z) = \sec xy - \tan z$

11. $f(x, y, y) = \cosh(xy - z)$

12. $f(x, y, z) = \frac{x - z}{y + 3x}$

13. Let $f(x, y) = xy^2$

(a)Calculate explicitly he difference $\Delta f - df$

(b)Verify your answer by calculating $\Delta f - df$ at the point $(1, 2)$, where $\Delta x = -0.01$ and $\Delta y = 0.03$.

$*$ 14. Repeat the steps of problem 13 for the function $f(x, y) = x^3y^2$.

In problems 15 -23, use the total differential to estimate the given number .

15. $\frac{3.01}{5.99}$

16. $19.8\sqrt{65}$

17. $\sqrt{35.6}\sqrt[3]{64.08}$

18. $(2.01)^4\ (3.04)^7\ - (2.01)\ (3.04)^9$

19. $\sqrt{\frac{5.02 - 3.96}{5.02 + 3.96}}$

20. $((4.95)^2\ + (7.02))^{\frac{1}{5}}$

21. $\frac{(3.02)(1.97)}{\sqrt{8.95}}$

22. $\sin(\frac{11\pi}{24})\ \cos\left(\frac{13\pi}{36}\right)$