THE CHAIN RULE

In this section we derive the chain rule for function s of two and three variables Let us recall the chain rule for the composition of two f unctions ofone variable :

Let $y = f(u)$ and $u = g(x)$ and assme that f and g are differentiable. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'\big(g(x)\big)g'(x) \quad (1)$$

If z =f (x, y) is a function of two variables , then there are two versions of the chain rule

Theorem 1 CHAIN RULE Let z =f (x, y) be differentiable and suppose that x =x (t) and y= y(t). Assume further that dx/dt and dy/dt exist and are continuous Then z can be written as a function of the parameter t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = f_x\frac{dx}{dt} + \frac{dy}{dt} \quad (2)$$

We can also write this result using our gradient If g(t) =x (t) I +y(t)j, then g'(t)= $(dx/dt)i + (dy/dt)j$, and (2) can be written as

$$\frac{d}{dt}f\big(x(t), y(t)\big) = (f\circ g)'(t) = [f(g(t))]' = \nabla f \cdot g'(t) \quad (3)$$

Theorem 2 CHAIN RULE Let z =f (x, y) be differentiable and suppose that x and y are function of the two variables r and s That is , x= x(r, s) and y =(r, s)

Suppose further that $\partial x/\partial r$, $\partial x/\partial s$, $\partial y/\partial r$ and $\partial y/\partial s$ all exist and are continuous . Then z can be written as a function of r and s, and

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad (4)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad (5)$$

We will leave the proofs of these theorems until the end this section .

EXAMPLE 1: Let $z = f(x, y) = xy^2$. Let $x = \cos t$ and $y = \sin t$ Calculate dz/dt

Solution.

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=y^2(-\sin t)+2xy(\cos t)$$

$=(\sin^2 t)(-\sin t)+2(\cos t\,(\sin t)\,(\cos t)$

$=2\sin t\cos^2 t-\sin^3 t$

We can calculate this result another way . Since $z=xy^2$ we have $z=(\cos t)(\sin^2 t)$ Then

$\frac{dz}{dt}=(\cos t)\,2(\sin t\,(\cos t)\;+\;(\sin^2 t)(-\sin t)$

$=2\sin t\cos^2 t-\sin^3 t$

EXAMPLE 2 Let $z=f(x,y)=\sin xy^2$ Suppose that $x=\frac{r}{s}$ and $y=e^{r-s}$ Calculate $\partial z/\partial r$ and $\partial z/\partial s$

Solution

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=(y^2\cos xy^2)\frac{1}{s}+(2xy\cos xy^2)e^{r-s}$$

$$=\frac{e^{2(r-s)}\cos[(\frac{r}{s})e^{2(r-s)}]}{s}+\frac{2r}{s}\left\{\cos[\frac{r}{s}e^{2(r-s)}]\right\}e^{2(r-s)}$$

And

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=(y^2\cos xy^2)\frac{-r}{s^2}+(2xy\cos xy^2)(-e^{r-s})$$

$$=\frac{-re^{2(r-s)}\cos[(\frac{r}{s})e^{2(r-s)}]}{s^2}+\frac{2r}{s}\left\{\cos[\frac{r}{s}e^{2(r-s)}]\right\}e^{2(r-s)}$$

The chain rules given in Theorem 1 and Theorem 2 can easily be extended to functions of three or more variables .

Theorem 1' Let w= f (x, y , z) be a differentiable function If $x=x(t)$, $y=y(t)$, $z=z(t)$, and if dx/dt, dy/dt, and dz/dt exist and are continuous, then

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{dw}{dz}\frac{dz}{dt}\quad(6)$$

Theorem 2' Let w = f (x, y , z) be a differentiable function and let x= x (r, s), y=y (r, s), and z=z (r, s) Then if all indicated partial derivatives exist and are continuous, we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \quad (7)$$

And

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \quad (8)$$

Theorem 3' Let w = f (x, y , z) be a differentiable function and let x= x (r, s, t), y=y (r, s), and z=z (r, s, t) Then if all indicated partial derivatives exist and are continuous, we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \quad (9)$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

PROBLEMS

In problems 1-11, use the chain rule to calculate dz/dt Check your answer by first writing z or w as a function of t and then differentiating.

1. $z = xy, x = e^t, y = e^{2t}$

2. $z = x^2 + y^2, x = \cos t, y = \sin t$

3. $z = \frac{y}{x}, x = t^2, y = t^3$

4. $z = e^x \sin y, x = \sqrt{t, y} = \sqrt[3]{t}$

5. $z = \tan^{-1}\frac{y}{x}, x = \cos 3t, y = \sin^{5t}$

6. $z = \sinh(x - 2y), x = 2t^2, y = t^2 + 1$

7. $w = x^2 + y^2, z^2, x = \cos t, y = \sin t, z = t$

8. $w = xy - yz + zx, x \neq e^t, y = e^{2t}, z = e^{3t}$

9. $w = \frac{x + y}{z}, x = t, y = t^2, z = t^3$

10. $w = \sin(x + 2y + 3z), x = \tan t, y = \sec t, z = t^5$

11. $w = \text{In}(2x - 3y + 4z), x = e^t, y = \text{Int}, z = \cos h\, t$

In problems 12-26, use the chain rule to calculate Check the indicated partial derivatives.

12. $z = xy; x = r + s; y = r - s; \partial z/\partial r$ and $\partial z/\partial s$

13. $z = x^2 + y^2; x = \frac{\cos(r+s); y = \sin(r-s); \quad \partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$

14. $z = \frac{y}{x}; x = e^r; y = \frac{e^s\ \partial z}{\partial r}$ and $\frac{\partial z}{\partial s};$

15. $z = \sin\frac{y}{x}; x = \frac{r}{s}; y = \frac{s}{r}; \frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$

16. $z = \frac{e^{x+y}}{e^{x-y}}; \ x = \text{In}\, rs; y = \text{In}\ \frac{r}{s}; \frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$

17. $z = x^2y^3; x = r - s^2; y = 2s + r; \ \partial z/\partial r$ and $\partial z/\partial s$

18. $w = x + y + z; x = rs; y = r + s; z = r - s; \ \partial w/\partial r$ and $\partial w/\partial s$

19. $w = \frac{xy}{z}; x = r, y = s, z = t; \frac{\partial w}{\partial r}, \frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$

20. $w = \frac{xy}{z}; x = r + s, y = t - r, z = s - 2t; \frac{\partial w}{\partial r}, \frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$

21. $w = \sin xyz; x = s^2r, y = r^2s, z = r - s; \ \frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$

22. $w = \sin h(x + 2y + 3z); x = \sqrt{r + s, y} = \sqrt[3]{s - t, z} = \frac{1}{r + t'}, \frac{\partial w}{\partial r}, \frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$

23. $w = xy^2 + yz^2; x = rst, y = \frac{rs}{t}, z = \frac{1}{rst}; \ \frac{\partial w}{\partial r}, \frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$

24. $w = \text{In}(x + 2y + 3z); x = rt^3 + s; y = t - s^5, z = e^{r+s}; \partial w/\partial r, \partial w/\partial s$ and $\partial w/\partial t$

25. $w = e^{\frac{xy}{z}}; x = r^2 + t^2, y = s^2 - t^2, z = r^2 + s^2; \frac{\partial w}{\partial r}, \frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$

$*$ 26. $u = xy + w^2 - z^3; x = t + r - q, y = q^2 + s^2 - t + r, z = \frac{qr + st}{r^2}, w = \frac{r - s}{t + q}; \frac{\partial u}{\partial r}, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial q}$

18 .7 TANGENT PLANES, NORMAL LINES, AND GRADIENTS

Let z= f(x, y) be a function of two variables . As we have seen, the graph of f is a surface in More general, the graph of the equation F(x , y, z) =0 is a surface in The surface F(x , y, z) =0 is called differentiable at appoint if all exist and are continuous at In a differentiable curve has a unique tangent line at each point . In a differentiable surface in has a unique tangent plane at each point at which are not all zero . We will formally define what we mean by a tangent plane to a surface after a bit, although it should be easy enough to visualize (see Figure 1). We note here that not every surface has a tangent plane at every point . For example, the cone has no tangent plane at the origin (see Figure 2) .

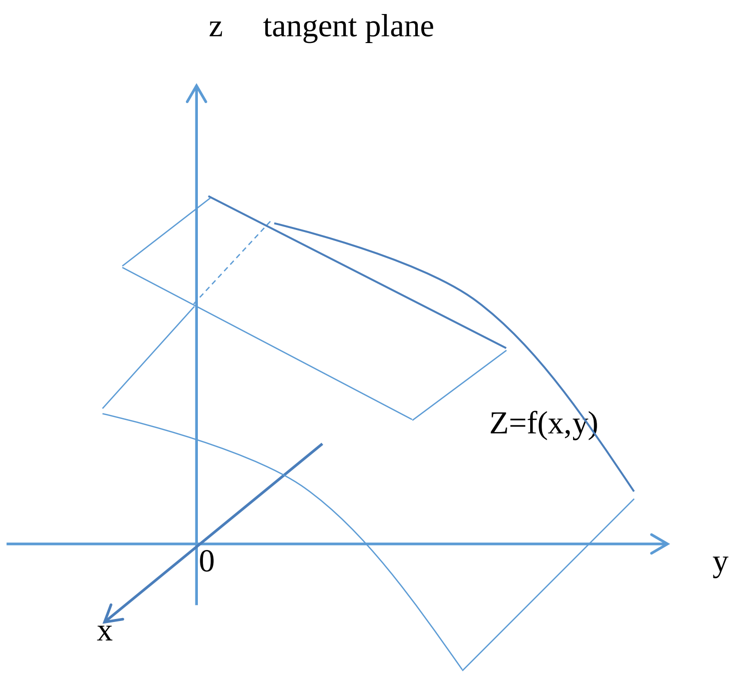

figure 1

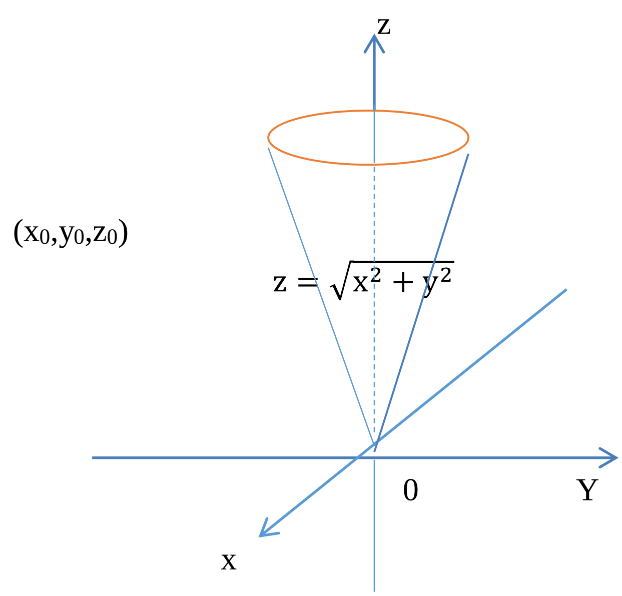

figure 2

Assume that the surface S given by F (x, y, z)=0 is differentiable . Let C be any curve lying on S. That is, C can be given parametrically by g (t)=x(t)I + y(t) j+ z(t) k.(Recall from Definition , the definition of a curve, F(x, y, z) can be written as a function of t, and from of the chain rule [equation (18. 6. 3)] we have

$$F'(t) = \nabla F . g'(t)$$

But since F (x(t), y(t),z (t))=0 for all t since (x(t), y(t),z (t)) is on S we see that F'(t)=0 for all t. But g '(t) is tangent to the curve number t. Thus (1) implies the following :

> The gradient of F at a point $x_0 = (x_0, y_0, z_0)$ on S is orthogonal to tangent vector at x_0 to any curve C remaining on S and passing through x_0

This statement is illustrated in Figure 3

surface

f(x,y,z)=0

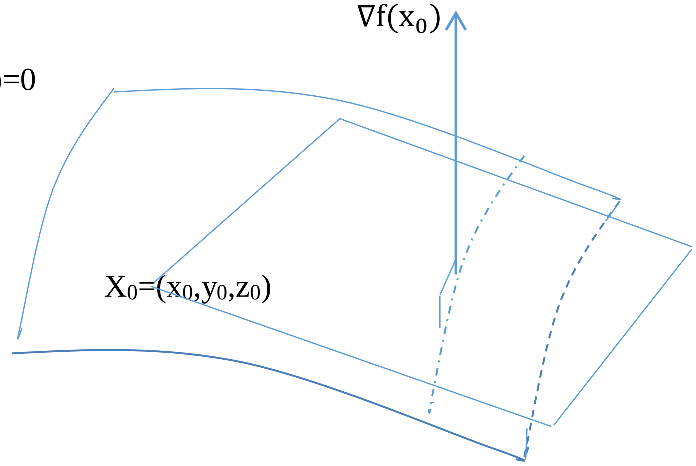

FIGURE 3

Thus is we think of all the vectors tangent to a surface at a point x_0 as constituting a plane, then $\nabla F\,(x_0)$ is normal vector to that plane . This motivates the following definition.

Definition 1 TANGENT PLANE AND NORMALLINE Let F be differentiable at $x_0 = (x_0, y_0, z_0)$ and let the surface S be defined by F(x , y, z)=0

(i)The tangent plane to S at$(x_0, y_0, z_0)$is the plane passing though the point $(x_0, y_0, z_0)$with normal vector $\nabla F\,(x_0)$

(ii)The normal line to S at x_0 is the line passing though x_0 having the same direction $\nabla F\,(x_0)$

EXAMPLE 1 Find the equation of the tangent plane and symmetric equations of the normal line to the ellipsoid $x^2 + \left(\frac{y^2}{4}\right) + \left(\frac{z^2}{9}\right) = 3$ at the point (1, 2, 3).

$x^2 + \left(\frac{y^2}{4}\right) + \left(\frac{z^2}{9}\right) - 3 = 0$ we haveSolution . Since $F(x,\ y, z) =$

$$\nabla F = \frac{\partial F}{\partial x}\,i + \frac{\partial f}{\partial y}\,j + \frac{\partial f}{\partial z}\,k = 2xi + \frac{y}{2}\,j + \frac{2z}{9}\,k$$

Then $\nabla F\,(1,2,3) = 2i + j + \frac{2}{3}\,k$, and the equation of the tangent plane is

$$2(x-1) + (y-2) + \frac{2}{3}\,(z-3) = 0,$$

Or

$$2x + y + \frac{2}{3}\,z = 6$$

The normal line is given by

$$\frac{x-1}{2} = y - 2 = \frac{3}{2}\,(z-3)$$

The situation is even simpler if we can write the surface in the form z= f(x , y).

That is , the surface is the graph of function of two variables. Then F(x, y, z)=

$$F_x = f_x\,, \quad F_y = f_y\,, \quad F_z = -1,$$

And the normal vector N to the tangent plane is

$$N = f_x\,(x_0, y_0)i + f_y\,(x_0, y_0)\,j - k \qquad (2)$$

REMARK . One interesting consequence of this fact is that if z = f(x, y)and if $\nabla f(x_0, y_0) = 0$, then the tangent plane to the surface at $(x_0, y_0, f(x_0, y_0)), N = (\partial f/\,\partial x)i + (\partial f/\,\partial y)j - k = -k$. is parallel to the xy-plane (i.e.,it is horizontal). This occurs because at Thus the z-axis is normal to the tangent plane.

EXAMPLE 2 Find the tangent plan and normal line to the surface z $=x^3y^5$ at the point (2, 1, 8)

Solution $N=\left(\frac{\partial f}{\partial x}\right) i + \left(\frac{\partial f}{\partial y}\right) j - k = 3x^2y^5 i + 5x^3y^4 j - k = 12i + 40j - k$ Then the tangent plane is given by

$$12(x-2) + 40(y-1) - (z-8) = 0,$$

Or

$$12x + 40y - z = 56$$

Symmetric equations of the normal line are

$$\frac{x-2}{12}=\frac{y-1}{40}=\frac{z-8}{-1}$$

We can write the equation of the tangent plane to a surface = z f(x ,y) so that it looks like the equation of the tangent line to a curve in This will further illustrate the connection between the derivative of a function of one variable and the gradient .Recall from Section 17. 5 that if p is appoint on a plane and N is normal vector, then if Q denotes any other point on the plane, the equation of the plane can be written

$$\overrightarrow{PQ}\,.\quad N=0\,. \tag{3}$$

In this case, since z = f (x ,y), ap0int on the surface takes the form (x , y , z)=(x , y , f(x, y)). Then since $N = f_x i + f_y j - k$, the equation of the tangent plane at $(x_0, y_0\ f(x_0, y_0))$ becomes , using (3),

$$0 = [(x , y , z) - (x_0, y_0 , z_0)]\ .\ (\ f_x + f_{y\,,} - 1\ )$$

$$= (x - x_0, y - y_0 , z - z_0)\,.\ (\ f_x + f_{y\,,} - 1)$$

$$= (x - x_0)\ f_x + (y - y_0)f_{y,} - (z - z_0) \tag{4}$$

We can rewrite (4) as

$$z = f\,(x_0, y_0) + (x - x_0)\ f_x + (y - y_0)f_{y,} \tag{5}$$

Denote (x_0, y_0) byx_0 and (x , y) by Then (5) can be written as

$$z = f\,(x_0) + (x - x_0)\ .\ \nabla f(x_0). \tag{6}$$

Recall the if y = f (x) is differentiable at x_0 then the equation of the tangent line to the curve at the point $(x_{0\,,}\ f\,(x_0))$ is given by

$$\frac{y - f\,(x_0)}{x - x_0} = f'(x_0)\,,$$

Or

$$y = f(x_0) + (x - x_0)f'(x_0). \tag{7}$$

This similarity between (6) and (7) illustrates quite vividly the important of the gradient vector of a function of several variables as the generalization of the derivative of a function of one variable .

PROBLEMS

In Problems 1-16 find the equation of the tangent plane and symmetric equations of the normal line to given surface at the given point .

1. $x^2 + y^2 + z^2 = 1; (1, 0, 0)$

2. $x^2 + y^2 + z^2 = 1; (0, 1, 0)$

3. $x^2 + y^2 + z^2 = 1; (0, 0, 1)$

4. $x^2 + y^2 + z^2 = 1; (1, 1, 1)$

5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3 ; (a , b, c)$

6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3 ; (-a , b, -c)$

7. $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = 6; (4, 1, 9)$

8. $ax + by + cz = d; \left(\frac{1}{a}, \frac{1}{b}, \frac{d-2}{c}\right)$

9. $xyz = 4; (1, 2, 2)$

10. $xy^2 + yz^2 + zx^2 = 1; (1, 1, 1)$

11. $4x^2 + y^2 + 5z^2 = 15; (3, 1, -2)$

12. $xe^y - ye^3 = 1; (1, 0, 0)$

13. $\sin xy - 2\cos yz = 0; \left(\frac{\pi}{2}, 1, \frac{\pi}{3}\right)$

14. $x^2 + y^2 + 4x + 2y + 8z = 7; (2, -3, -1)$

15. $e^{xyz} = 5; (1, 1, \text{In } 5)$

16. $\sqrt{\frac{x+y}{z-1}} = 1; (1, 1, 3)$

In Problems 7-24, write the equation of the tangent plane in the form (6) and find the symmetric equations of the normal line to given surface

17. $z = xy^2; (1, 1, 1)$

18. $z = \text{In}(x - 2y); (3, 1, 0)$

19. $z = \sin(2x + 5y); \left(\frac{\pi}{8}, \frac{\pi}{20}, 1\right)$

20. $z = \sqrt{\frac{x+y}{x-y}}; (5, 4, 3)$

21. $z = \tan^{-1}\frac{y}{x} ; \left(-2, 2, -\frac{\pi}{4}\right)$

22. $z = \sinh xy^2; (0, 3, 0)$

23. $z = \sec(x - y); \left(\frac{\pi}{2}, \frac{\pi}{6}, 2\right)$

24. $z = e^x \cos y + e^y \cos x ; \left(\frac{\pi}{2}, 0 , e^{\frac{\pi}{2}}\right)$

$*$ 25. Find the two points of intersection of the surface $z = x^2 + y^2$ and the line

$$\frac{x-3}{1} = \frac{y+1}{-1} = \frac{z+2}{-2} .$$