DIFFERENTILITY AND THE GRADIENT

In this section we discuss the notion of the differentiability of a function of several variables. There are several ways to introduce this subject and the way we have chosen is designed to illustrate the great similarities between differentiation of functions of one variable and differentiation of functions of several variables .

We begin with a function of one variables,

Y =f (x).

If f is differentiable , then

F' (x)= $\frac{dy}{dx} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x}$

Then if we define the new function $\in (\Delta x)$ by

$$\in (\Delta x) = \frac{\Delta y}{dx} - f'(x),$$

We have

$$\lim_{\Delta x \to o} \in (\Delta x) = \lim_{\Delta x \to o} \left(\frac{\Delta y}{\Delta x} - f'(x) \right) = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x} - f'(x)$$

= f' (x) - f ' (x) = 0 .

Multiplying both sides of (2) by Δx and rearranging terms, we obtain

$$\Delta y = f'(x)\Delta x + \in (\Delta x) \ \Delta x .$$

Note the here Δy depends on both Δx and x. Finally, since $\Delta y = f'(x + \Delta x) - f(x)$, we obtain

$$f'(x + \Delta x) - f(x) = f'(x)\Delta x + \in (\Delta x) \ \Delta x .$$

Why did we do all this? We did so in order to be able to state the following alternative definition of differentiability of a function f of one variable.

Definition 1 ALTERNATIVE DEFINITON OF DIFFRENTIABILITY OF A FUNCTION OF ONE VARIABLE Let f be function of one variable . Then f is differentiable at a number x if there is function f' (x) and a function $g(\Delta x)$ such that

$$f'(x + \Delta x) - f(x) = f'(x)\Delta x + g(\Delta x),$$

Where $\lim_{\Delta x \to o[} g(\Delta x) / (\Delta x)] = 0$.

We will soon show how the definition (5) can be extended to a function of two or more variables. First, we give a definition

Definition 2 DIFFRENTIABILITY OF A FUNCTION OF TOW VARIABLES: Let f be a real –valued

Function of two variables that is defied in a neighborhood of a point (x, y)and such that $f_x(x, y)$and $f_y(x, y)$ exist.Then f is differentiable a (x, y) if there exist function$\in_1$ (Δx , Δy)and $\in_2$ (Δx , Δy) such that

$$f\ (x+\ \Delta x\ ,y+\ \Delta y\ )-f\ (x\ ,y\ )=\ f_x(x,y\ )\Delta y+\in_1(\Delta x\ ,\Delta y\ )+\in_2(\Delta x\ ,\Delta y\ )\ \Delta y,$$

Where

$$\lim_{(\Delta x,\Delta y)\to(o,0)}\in_1(\Delta x\ ,\Delta y\ )=0\ and\quad \lim_{(\Delta x,\Delta y)\to(o,0)}\in_2(\Delta x\ ,\Delta y\ )=0.$$

DIFFERENTI A BILTY AND THE GRADIENT
In this section we discuss the notion of the differentiability of a function of several variables. There are several ways to introduce this subject and the way we have chosen is designed to illustrate the great similarities between differentiation of functions of several variables.

We being with a function of one variables.

Y = f(x)

If f is differentiable , then

$$f'(x)=\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \qquad (1)$$

Then if we define the new function $\in(\Delta x)$ by

$$\in(\Delta x)=\frac{\Delta y}{\Delta x}-f'(x), \qquad (2)$$

We have

$$\lim_{\Delta x\to 0}\in(\Delta x)=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}-f'(x)\right)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}-f'(x)=f'(x)-f'(x)=0(3)$$

Multiplying both sides of (2) by Δx and rearranging terms , we obtain

$$\Delta y = f'(x)\Delta x + \in (\Delta x)\, \Delta x$$

Note that hereΔy depends on both Δx and x Finally ,since $\Delta y = f(x+\Delta x) - f(x)$, we obtain

$$f(x+\Delta x) - f(x) = f'(x)\Delta x + +\in (\Delta x)\, \Delta x \qquad (4)$$

Why did we do all this ? We did so in order to be able to state the following alternative definition of differentiability of a function f of one variable

Definition 1 ALTRNATIVE DEFINTION OF DIFFERENTIABILITY OF A FUNCTION OF ONEVARIABLE Let f be a function f of one variable Then f is differentiable at a number x if there is a function f' (x) and a function g (Δx) such that

$$f(x+\Delta x) - f(x) = f'(x)\Delta x + g(\Delta x) (5)$$

Where $\lim\limits_{\Delta x \to 0} \left[\frac{g(\Delta x)}{\Delta x}\right) = 0$

We will soon show how the definition (5) can be extended to two or more variables . First , we give a definition .

Definition 2 DIFFERENTIABILITY OF A FUNCTION OF TOW VARIABLE S Let f be areal –valued function f of two variable s that $f_x(x,y)$and $f_y(x,y)$ exist Then f is differentiable at (x, y) if there exist function f' (x) and a functions$\in_1(\Delta x, \Delta y)$and $\in_2(\Delta x, \Delta y)$such that

$$\begin{aligned} &f(x+\Delta x, y+\Delta y) - f(x,y) \\ &\qquad = f_x(x,y)\Delta x + f_y(x,y)\Delta y + \in_1(\Delta x, \Delta y)\Delta x \\ &\qquad + \in_2(\Delta x, \Delta y)\Delta y \end{aligned} \qquad (6)$$

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \in_1(\Delta x, \Delta y) = 0 \quad and \quad \lim_{(\Delta x, \Delta y)\to(0,0)} \in_2(\Delta x, \Delta y) = 0 \qquad (7)$$

EXAMPLE 1 Let f (x, y) = xy . Show that f is differentiable at every point (x , y) in R^2

Solution

$$\begin{aligned} f(x+\Delta x, y+\Delta y) - f(x,y) &= (x+\Delta x)(y+\Delta y) - xy = xy + y\Delta x + x\Delta y + \Delta x\Delta y - xy \\ &= y\Delta x + x\Delta y + \Delta x\Delta y \end{aligned}$$

Now $f_x = y$ and $f_y = x$ so we have

$$f\,(x+\Delta x, y+\Delta y) - f\,(x, y\,) = f_x(x, y)\Delta x + \quad f_y(x, y)\Delta y + +\Delta y\Delta x + 0.\,\Delta y$$

Setting $\in_1(\Delta x\,,\Delta y) = \Delta y$ and $\in_2(\Delta x\,,\Delta y) = 0$ we see that

$$\lim_{(\Delta x\,,\Delta y)\to(0\,,0)} \in_1(\Delta x\,,\Delta y) = \lim_{(\Delta x\,,\Delta y)\to(0\,,0)} \in_2(\Delta x\,,\Delta y) = 0$$

This result shows that f (x, y) = xy is differentiable at every point in R^2

We now rewrite our definition of differentiability in a more compact from Since a point(x, y) is a vector inR^2 we will write (as we have done before) x =(x , y) .Then if z =f (x , y) we can simply write

Z = f (x) .

Similarly , if w = f (x, y , z)we may write

W= f (x) ,

Where x is the vector (x, y , z). With this notation we may use the symbol Δx to denote the vector$(\Delta x\,,\Delta y)$in R^2 or$(\Delta x\,,\Delta y\,,\Delta z)$ inR^2

Next , we write

$g(\Delta x) = \in_1(\Delta x\,,\Delta y)\Delta x + \in_2(\Delta x\,,\Delta y)\,\Delta y(8)$

Note that$(\Delta x\,,\Delta y) \to (\,0\,,0\,)$ can be written in the compact form $\Delta x \to 0$ .Then if the conditions (7) hold, we see that

$$\downarrow |\Delta x| = \sqrt{\Delta x^2 + \Delta y}$$

$$\lim_{\Delta x\to 0}\frac{|\,g\,(\Delta x)|}{|\Delta x|} \le \lim_{\Delta x\to 0}|\in_1(\Delta x\,,\Delta y)|\frac{|\Delta x|}{\sqrt{\Delta x^2+\Delta y^2}} + \lim_{\Delta x\to 0}|\in_2(\Delta x\,,\Delta y)|\frac{|\Delta x|}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$\frac{|\Delta x|}{\downarrow\sqrt{\Delta x^2+\Delta y^2}} \le 1$$

$$\le \lim_{\Delta x\to 0}|\in_1(\Delta x\,,\Delta y)| + \lim_{\Delta x\to 0}|\in_2(\Delta x\,,\Delta y)| = 0 + 0 = 0$$

Finally , we have the following important definition

Definition 3 THE GRADIENT Let f be a function f of two variables such that f_x, and f_y, exist at a point x =(x, y) Then the gradient of at f at x , denoted $\nabla f(x)$, is given by

$$\nabla f(x) = f_x(x, y)i + f_y(x, y)j \qquad (9)$$

Example 1, f (x, y) =xy $f_x = y$, and $f_y = x$, so that

Using this new notation , we observe that

$$\nabla f(x).\Delta x = (f_x\, i + f_y j).(\Delta x i + \Delta y\, j) = f_x(x, y)\Delta x + f_y(x, y)\Delta y$$

Also ,

$$f(x + \Delta x, y + \Delta y) = f(x + \Delta x).$$

Thus we have the following definition , which is implied Definition 2 .

Definition 4 DIFFERENTIABILITY Let f be a function of two variables that is definition in a neighborhood of a point x = (x, y). Let $\Delta x = (\Delta x, \Delta y)$. If $f_x(x, y)$ and $f_y(x, y)$ exist , then f is differentiable at x if there is a function g such that

$$f(x + \Delta x) - f(x) = \nabla f(x).\ \Delta x + g(\Delta x), \qquad (10)$$

Where

$$\lim_{(\Delta x \to 0)} \frac{g(\Delta x)}{|\Delta x|} = 0 \qquad (11)$$

Theorem 1: Let f , f_x , and f_y bedefined and con tenuous in a neighborhood of x = (x , y) Then f is differentiable at x

EXAMPLE 3 Let z =f (x, y)$xy^2 + e^{x^2y^3}$ Show that f is differentiable and calculate ∇f. Find $\nabla f(1, 1)$

Solution . $\frac{\partial f}{\partial x} = y^2 \cos xy^2 + 2xy^3 e^{x2y^3}$ and $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} = y^2 \cos xy^2 + 2xy^3 e^{x2y^3}$

Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuo us f is differentiable and

$$\Delta f(x, y) = (y^2 \cos xy^2 + 2xy^3 e^{x2y^3})i + (2xy \cos xy^2 + 3x^2y^2 e^{x2y^3})\, j$$

At (1 ,1),$\nabla f(1, 1) = (\cos 1 + 2e)i + (2\cos 1 + 3e)j$

we showed that the existence of all of its partial derivatives at a point dose not ensure that a function is continuous at that point . However, differentiability (according to Definition 4) does ensure continuity .

Theorem 2 : If f is differentiable at $x_0 = (x_0, y_0)$ then f is continuous at x_0

Proof We must show that $\lim_{\Delta x \to 0} f(x) = f(x_0)$ But if we defineΔx by $\Delta x = x - x_0$ this is the same as showing that

$$\lim_{\Delta x \to 0} f(x_0 + \Delta x) = f(x_0)(12)$$

Since f is differentiable at x_0

$$f(x_0 + \Delta x) - f(x_0) = \nabla f(x_0).\Delta x + g(\Delta x). \tag{13}$$

But as$\Delta x \to 0$, both terms on the right –hand side of (13) approach zero, so

$$\lim_{\Delta x \to 0}[f(x_0 + \Delta x) - f(x_0)] = 0,$$

Which means that (12) holds and the theorem is proved.

The converse to this theorem is false, as it is in one –variable calculus. That is, there are functions that are continuous, but not differentiable, at a given point . For example, the function

$$f(x, y) = \sqrt[3]{x} + \sqrt[3]{y}$$

Is continuous at any point (x, y) inR^2 But

$$\nabla f(x, y) = \frac{1}{3x^{2/3}}\, i + \frac{1}{3y^{2/3}}\, j,$$

So f not differentiable at any point (x, y) for which either x or y is zero . That is , is not defined on the x – and y- axes Hence f is not differentiable along these axes .

we showed that

$(f + g)' = f' + g'$ and $(af)' = af'$;

that is , the derivative of the sum of two functions is the sum of the derivatives of the two functions and the derivative of a scalar multiple of a two functions is the scalar times the derivative of the function . These results can be extended to the gradient vector .

Theorem 3 Let f and g be differentiable in a neighborhood of x = (x, y) . Then for every scalar a, ,a.f and f +g are differentiable at x, and

$(i)\nabla(af) = a\nabla f, and$

$(ii)\nabla(f+g) = \nabla f + \nabla g$.

Proof

(i)Form the definition of differentiability (Definition 4), there is a function $h_1(\Delta x)$such that

$$f(x+\Delta x) - f(x) = \nabla f(x).\Delta x + h_1(\Delta x),$$

Where $\lim_{\Delta x\to 0}[(h\_1 (\Delta x))/|\Delta x|] = 0$ Thus $af(x+\Delta x) - af(x) = a\nabla f(x).\Delta x + a\,h_1(\Delta x)$, and

$$\lim_{\Delta x\to 0}\frac{ah_1(\Delta x)}{|\Delta x|} = a\lim_{\Delta x\to 0}\frac{h_1(\Delta x)}{|\Delta x|} = a0 = 0.$$

But

$$a\,\frac{\partial f}{\partial x} = \frac{\partial\,\partial(af)}{\partial x}$$

$$a\nabla f(x) = a\left(f_x i + f_y j\right) = (af)_x i + (af)_y j = \nabla(af)$$

Thus

$af(x+\Delta x) - af(x) = \nabla af(x)\ .\Delta x + \ ah_1(\Delta x),$

Which shows that af is differentiable and$\nabla(af) = a\nabla f$

(ii)As above, there is a function $h_2(\Delta x)$such that $g(x,\Delta x) - g(x) = \nabla g(x).\Delta x + h_2(\Delta x)$, where

$$(f+g)(x+\Delta x) - (f+g)(x) = [\,f(x+\Delta x) + g(x+\Delta x)] - [f(x) + g(x)]$$

$$= [f(x+\Delta x) - f(x)] + [g(x) + \Delta x) - g(x)]$$

$$= \nabla f(x)\,.\Delta x + h_1(\Delta x) + \nabla g(x).\ \Delta x + h_2(\Delta x)$$

$$= [\nabla f(x) + \nabla g(x)].\Delta x + [h_1(\Delta x) + h_2(\Delta x)],$$

Where

$$\lim_{\Delta x \to 0} \frac{[h_1(\Delta x) + h_2(\Delta x)]}{|\Delta x|} = 0.$$

To complete the proof, we observe that

$$\nabla f(x) + \nabla g(x) = (f_x i + f_y j) + (gi + g_y j) = (g_x + g_y)i + (g_x + g_y)j$$

$$\frac{\partial}{\partial x}(f+g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}$$

$$= (f+g)_x i + (f+g)_y j = \nabla(f+g).$$

Thus $f+g$ is differentiable and $\nabla(f+g) = \nabla f + \nabla g$.

REMARK. Any function that satisfies conditions (i) and (ii) of Theorem 3 is called a linear mapping or linear operator . Linear operator play an extremely important role in advanced mathematics

All the definitions and theorems in this section hold for functions of three or more variables. We give the equivalent results for functions of three variables below .

Definition 5 THE GRADIENT Let f be scalar function of three variables that f_x, f_y, and f_z exist at a point x = (x, y, z) Then the gradient of f at x, denoted $\nabla f(x)$, is given by the vector

$$\boxed{\nabla f(x) = f_x(x,y,z)i + f_y(x,y,z)j + f_z(x,y,z)k. \qquad (14)}$$

Definition 6 DIFFERENTIABILITY Let f be a function of three variables that is defined in a neighborhood of x = (x, y, z) ,and let $\Delta x = (\Delta x, \Delta y, \Delta z)$. If $f_x(x,y,z) f_y(x,y,z)$, and $f_z(x,y,z)$ exist then f is differentiable at x if there is a function g such that

$$\boxed{f(x+\Delta x) - f(x) = \nabla f \,.\, \Delta x + g(\Delta x)}$$

Where

$$\boxed{\lim_{|\Delta x| \to 0} \frac{g(\Delta x)}{|\Delta x|} = 0}$$

Equivalently , we can write

$$\boxed{\begin{aligned} &f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z) \\ &\qquad = f_x(x,y,z)\Delta x + f_y(x,y,z)\Delta y + f_z(x,y,z)\Delta z + g(\Delta x, \Delta y, \Delta z), \end{aligned}}$$

Where

$$\lim_{(\Delta x,\Delta y,\Delta z)\to(0,0,0)} \frac{g(\Delta x,\Delta y,\Delta z)}{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}} = 0$$

Theorem 1' If f, f_x ,f_y and f_z exist and are continuous in a neighborhood of x = (x, y, z)then f is differentiable at x

Theorem 2' Let f be function of three variables that is differentiable at x_0 Then f is continuous at x_0

EXAMPLE 4 Let $f(x, y, z) = xy^2z^3$ show that f is differentiable at any point x_0 calculate ∇f, and find $\nabla f(3,-1,2)$

Solution $\frac{\partial f}{\partial x} = y^2z^3, \frac{\partial f}{\partial y} = 2xyz^3$ and $\frac{\partial f}{\partial z} = 3\,xy^2z^2$ Since $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are all continuous , we know that f is differentiable and that

$\nabla f = y^2z^3 i + 2xyz^3 j + 3\,xy^2z^2 k$

And

$\nabla f(3,-1,2) = 8i - 48j + 36k$.

Theorem 3' Let f and g be differentiable in a neighborhood of x = (x, y, z)Then for any scalar a , af and f+ g are differentiable at x , and

$$(i)\nabla(af) = a\nabla f\,, \quad and$$

$$(ii)\nabla(f+g) = \nabla f + \nabla g$$

We conclude this section with a proof of Theorem 1 . The proof of Theorem 1' is similar

proof of Theorem 1 . We begin by restating the mean value theorem for a function f of one variable .

Mean Value Theorem Let f be continuous on $[a, b\,]$anddifferentiableon(a, b)Thenthereisanumbercin$(a,\ b)$such that

F (b) - f(a)=f '(c)(b –a).

Now we have assumed that f,f_x, andf_y, are all continuous in a neighborhood N of x = (x, y) ChooseΔx so small that x+Δx N . Then

,y +Δy) − f (x, y)Δf(x) = f(x + Δx

$$\overbrace{[f\ (x+\Delta x, y+\Delta y) - f(x+\Delta x\ , y)] + [f\ (x+\Delta x, y) - f\ (x, y\ )]}^{\text{This lerm was added and subiracted}} \quad (15)$$

If x + is fixed , then f (x + Δx, y) is a function of y that
is. Hence by the mean value theorem there is a number c_2 between y and y + Δy such that

$$f\ (x+\Delta x, y+\ \Delta y) - f\ (x+\Delta x, y) = f_y(x+\Delta x, c_2)[(y+\Delta y) - y\ ]$$
$$= f_y(x+\Delta x, c_2)\Delta y \quad (16)$$

Similarly, with fixed , f (x, y) is a function of x only , and we obtain

$$f\ (x+\Delta x, y+\ y) - f(x+y) = f_x(c_1\ , y)\ \Delta x, (17)$$

Where c_1 is between x and x +Δx Thus using (16) and (17) in (15) , we have

$$\Delta f\ (x) = \ f_x(c_1\ , y)\ \Delta x + f_y(x+\Delta x, c_2)\Delta y\ . \quad (18)$$

Now bothf$_x$ and $f_y$arecontinuousatx = (x, y)sosince $c_1$isbetweenxandx + Δx and $c_2$isbetweeny + Δy, weobtain

$$\lim_{\Delta x \to 0} f_x(c_1, y) = f_x(c_1, y) = f_x(x)(19)$$

And

$$\lim_{\Delta x \to 0} f_y(x+\Delta x, c_2) = f_y(c_1, y) = f_y(x)(20)$$

Let

$$\in_1\ (\Delta x) = f_x(c_1, y) - f_x(c_1, y)\ . \quad (21)$$

From (19) it follows that

$$\lim_{\Delta x \to 0} \in_1\ (\Delta x) = 0 \quad (22)$$

Similarly , if

$$\in_1\ (\Delta x) = f_y(x+\Delta x, c_2) - f_y(x, y), \quad (23)$$

Then

$$\lim_{|\Delta x|\to 0} \epsilon_2\ (\Delta x) = 0(24)$$

Now define

$$g(\Delta x) = \epsilon_1\ (\Delta x)\ \Delta x + \epsilon_2\ (\Delta x)\Delta y \qquad (25)$$

From (22) and (24) it follows that

$$\lim_{|\Delta x|\to 0} \frac{g(\Delta x)}{|\Delta x|} = 0\ . \qquad (26)$$

Finally , since

$$f_x(c_1, y) - f_x(c_1, y) + \epsilon_1\ (\Delta x) \qquad \text{from}(21) \qquad (27)$$

And

$$f_y(x + \Delta x, c_2) = f_y(x, y) + \epsilon_2\ (\Delta x), \qquad \text{from}(23) \qquad (28)$$

We may substitute (27) and (28) into (18) to obtain

$$\Delta f(x) = f(x + \Delta x) - f(x) = [f_x(x) + \epsilon_2\ (\Delta x)]\Delta x + \left[f_y(x) + \epsilon_2\ (\Delta x)\right]\Delta y$$
$$= f_x(x)\Delta x + f_y(x)\Delta y\ + g(\Delta x) = \left(f_x i + f_y j\right).\ (\Delta x) + g(\Delta x),$$

Where

$$\lim_{|\Delta x|\to 0} = [g(\Delta x)/|\Delta x|\ ] \to 0, \text{and the proof is (at last)complete .}$$

PROBLEMS

1. Let $f\ (x, y) = x^2y^2$ Show , by using Definition 2 , that f is differentiable at any point in R^2

2. Let $f\ (x, y) = x^2y^2$Show , by using Definition 2 , that f is differentiable at any point in R^2

3 . Let $f\ (x, y) =$ be any polynomial in the variables x and y . Showthat f is differentiable

In problems 4- 24 calculate the g gradient of the given function If a point is also given, evaluate the gradient at that point .

4. $f\ (x, y) = y\ (x + y)^2$ 5. $f\ (x, y) = e^{\sqrt{xy}}; (1, 1)$

6 . $f\ (x, y) = \cos(x - y); \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$ 7. $f\ (x, y) = \text{In}(2x - y + 1)$

8 . $f(x,y) = \sqrt{x^2 + y^3}$ 9. $f(x,y) = \tan^{-1}\frac{y}{x}$; (3,3)

10. $f(x,y) = y\tan(y - x)$ 11 . $f(x,y) = x^2 \sinh y$

12 . $f(x,y) = \sec(x + 3y)$; (0, 1) 13. $f(x,y) = \frac{x - y}{x + y}$; (3, 1)

14. $f(x,y) = \frac{x^2 - y^2}{x^2 - y^2}$ 15. $f(x,y) = \frac{e^{x2} - e^{-y2}}{3y}$

16. $f(x,y,z) = xyz$; (1,2, 3) 17 . $f(x,y,z) = \sin x \cos y \tan z$; $(\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3})$

18 . $f(x,y,z) = \frac{x^2 - y^2 + z^2}{3xy}$; (1, 2,0) 19 . $f(x,y,z) = x\ln y - z\ln x$

20 . $f(x,y,z) = xy^2 + y^2z^3$; (2, 3, −1) 21. $f(x,y,z) = (y - z)e^{x+2y+3z}$; (−4, −1, 3)

22. $f(x,y,z) = x\sin y \ln z$; (1, 0, 1) 23 . $f(x,y,z) = \frac{x - z}{\sqrt{1 - y^2 + x^2}}$; (0, 0, 1)

24. $f(x,y,z) = x\cosh z - y\sin x$

25. Show that if f and g are differentiablefunction of three variables , then $\nabla(f + g) = \nabla f + \nabla g$

26. Show that if f and g are differentiablefunction of three variables , then fg is differentiable and

$\nabla(fg) = f(\nabla g) + g(\nabla f)$.

∗ 27. Show that $\nabla f = 0$ if and 0nly if f is constant

∗ 28. Show that $\nabla f = \nabla g$, then there is a constant c for which $f(x,y) = g(x,y) + c$ [Hint : Use the result of Problem 27.]

∗ 29 . What is the most general function f such that $\nabla f(x) = x$ for every x in R^2 ?

∗ 30 . Let
$$f(x,y) = \begin{cases} (x^2 + y^2)\sin\frac{1}{\sqrt{x^2 + y^2}} & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

(a) Calculate $f_x(0,0)$ and $f_y(0,0)$

(b)Explain why f_x and $f_y$are not continuous at $(0, 0)$

(c) Show that f is differentiable at(0 ,0)

31. Suppose that f is differentiable function of one variable and g is a differentiable function of three variables. Show that $f°g$ is differentiable and $\nabla f\,°g = f'(g)\nabla g$