HIGER – ORDER PARTIAL DERIVATIVES

We have seen that if y = f(x), then

$$y' = \frac{df}{dx} \text{ and } y'' = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

That is ,the second derivative of f is the derivative of the first derivative of f. , if z = f (x, y), then we can differentiate each of the two "first" partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ with respect to both x and y to obtain four second partial derivatives as follows :

Definition 1: SECOND PARTIAL DERIVATIVES

(i) Differentiate twice with respect to x :

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad (1)$$

(ii) Differentiate first with respect to x and then with respect to y :

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial y\, \partial x} = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad (2)$$

(iii) Differentiate first with respect to y and then with respect to x :

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial x\, \partial y} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad (3)$$

(iv) Differentiate twice with respect to y :

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \quad (4)$$

REMARK1.Thederivatives $\partial^2 f / \partial x\, \partial y$ and $\partial^2 f / \partial y\, \partial x$ arecalled the mixed second partialsl.

REMARK2. It is much easier to denote the second partials by f_{xx}, f_{xx}, f_{yx} and f_{yy}, We

Will there f are use this notation for the remainder of this section. Note that the symbol f_{xy} indicates that we differentiate first with respect to y .

EXAMPLE 1 Let $z = f(x,y) = x^3y^2 - xy^5$ .Calculate the four second partial derivatives .

Solution. We have $f_x = 3x^2y^2 - y^5$ and $f_y = 2x^3y - 5xy^4$.

(a) $f_{xx} = \frac{\partial}{\partial x}(f_x) = 6xy^2$

(b) $f_{xy} = \frac{\partial}{\partial y}(f_x) = 6x^2 y - 5y^2$

(c) $f_{xy} = \frac{\partial}{\partial x}(f_y) = 6x^2 y - 5y^2$

(d) $f_{yy} = \frac{\partial}{\partial y}(f_y) = 2x^3 - 20xy^3$

In Example 1 we saw that$f_{xy} = f_{yx}$ This result is no accident ,as we see by the following theorem whose proof can be found in any intermediate calculus text .t

Theorem 1 : Suppose that f, f_x, f_y, f_{xy} and f_{yx} are all continuous at (x_0, y_0) Then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0) \qquad (5)$$

This result is often referred to as the equality of mixed partials ≠0

The definition of second partial derivatives and the theorem on the equality of mixed partials are easily extended to functions of three variables. If w =(x ,y , z), then we have the nine second partial derivatives (assuming that they exist) :

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial y\ \partial x} = f_{xy}, \quad \frac{\partial^2 f}{\partial z\ \partial x} = f_{xz},$$

$$\frac{\partial^2 f}{\partial x\ \partial y} = f_{yx}, \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}, \quad \frac{\partial^2 f}{\partial z\ \partial y} = f_{yz}$$

$$\frac{\partial^2 f}{\partial x\ \partial z} = f_{zx}, \quad \frac{\partial^2 f}{\partial y\ \partial z} = f_{zy}, \quad \frac{\partial^2 f}{\partial z^2} = f_{zz}.$$

Theorem2 If f, f_x, f_y, f_z and f_{yx} and all six mixed partial are continuous at a point (x_0, y_0, z_0)then at a point This theorem was first stated by Euler in a 1734 paper devoted to a problem in hydrodynamics

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \quad f_{yz} = f_{zy}.$$

EXAMPLE 2 : Let $f(x, y, z) = xy^3 - zx^5 + x^2yz$ be a function , Calculate all for nine second partial derivatives and show that all three pairs of mixed partials are equal

Solution : We have

$f_x = y^3 - 5zx^4 + 2xyz$,

$f_y = 3xy^2 + x^2z$,

and

$f_z = -x^5 + x^2y$

Then

$f_{xx} = -20zx^3 + 2yz$, $f_{yy} = 6xy$, $f_{zz} = 0$,

$$f_{xy} = \frac{\partial}{\partial y}(y^3 - 5zx^4 + 2xyz) = 3y^2 + 2xz ,$$

$$f_{yx} = \frac{\partial}{\partial x}(3xy^2 + x^2z) = 3y^2 + 2xz ,$$

$$f_{xz} = \frac{\partial}{\partial z}(y^3 - 5zx^4 + 2xyz) = -5x^4 + 2xy ,$$

$$f_{zx} = \frac{\partial}{\partial x}(-x^5 + x^2 y) = -5x^4 + 2xy ,$$

$$f_{yz} = \frac{\partial}{\partial z}(3xy^2 + x^2 z) = x^2,$$

$$f_{zy} = \frac{\partial}{\partial y}(-x^5 + x^2 y) = x^2$$

We conclude this section by pointing out that we can easily define partial derivatives of orders higher than two . For example,

$$f_{zyx} = \frac{\partial^3 f}{\partial x\, \partial y\, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\, \partial z}\right) = \frac{\partial}{\partial x}(f_{zy})$$

EXAMPLE 3 Calculate and for the function of Example 2 .

Solution We easily obtain the three third partial derivatives:

$$f_{xxx} = \frac{\partial}{\partial x}(f_{xx}) = \frac{\partial}{\partial x}(20zx^3 + 2yz) = -60zx^2$$

$$f_{zzy} = \frac{\partial}{\partial y}(f_{xz}) = \frac{\partial}{\partial y}(5x^4 + 2yz) = 2x$$

$$f_{yxz} = \frac{\partial}{\partial z}\left(f_{yx}\right) = \frac{\partial}{\partial z}(3y^2 + 2xz) = 2x$$

Note that $f_{xzy} = f_{yxz}$This again is no accident and follows from the generalization of Theorem 2 to mixed third partial derivatives. Finally, the fourth partial derivative $f_{yxzx}$is given by

$$f_{yxzx} = \frac{\partial}{\partial x}\left(f_{yxz}\right) = \frac{\partial}{\partial x}(2x) = 2.$$

<u>PROBLEMS</u>

In problems 1-12 , calculate the four second partial derivatives and show that the mixed partials are equal .

1 . $f(x, y) = x^2 y$.

. $f(x, y) = xy^2$ y.2

3 . $f(x, y) = 3e^{xy3}$

4. $f(x, y) = \sin(x^2 + y^3)$

5 . $f(x, y) = \frac{4x}{y^5}$

6 . $f(x, y) = e^y \tan_x$.

7 . $f(x, y) = \text{In}(x^3y^5 - 2)$

8 . $f(x, y) = \sqrt{xy + 2y^3}$

9. f (x ,y) = (x + 5y sin x)اكتبالمعادلةهنا.

10 . f (x , y) = sinh (2x - y)

11 . $f(x, y) = \sin^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$

12 . f (x , y) =sec x y

In Problems 13 -21 , calculate the nine second partial derivatives and show that the three pairs of mixed partials are equal

13 . $f(x, y, z) = xyz$

14 . $f(x, y, z) = x^2y^3z^4$

15 . $f(x, y, z) = \frac{x+y}{z}$

16 . $f(x, y, z) = \sin(x + 2y + z^2)$

17. $f(x, y, z) = \tan^{-1}\frac{xz}{y}$

18 . $f(x, y, z) = \cos xyz$

19 . $f(x, y, z) = e^{3xy} \cos z$

20 . $f(x, y, z) = \ln(xy + z)$

21 . $f(x, y, z) = \cosh\sqrt{x + yz}$

22 . How many third partial derivatives are there for a function of (a) two variables; (b) three variables?

23 . How many fourth partial derivatives are there for a function of (a) two variables; (b) three variables?

24 . How many nth partial derivatives are there for a function of (a) two variables; (b) three variables ?

In Problems 25 -30 , calculate the given partial derivative

25 . $f(x, y) = x^2y^3 + 2y$: f_{xyx}

26 . $f(x, y) = \sin(2xy^4)$; f_{xyt}

27 . $f(x, y) = \text{In}(3x - 2y)$; f_{yxy}

28 . $f(x, y, z) = x^2y + y^2z - 3\sqrt{xz}$; f_{xyz}

29. $f(x, y, z) = \cos(x + 2y + 3z)$; f_{zzx}

30 $f(x, y, z) = e^{xy}\sin z$; f_{zxyx} .