Ex: Use Tayler polynomial with $n=8$ to approximate $\int_{-1}^{1} \cos x^2 \, dx$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$\cos x^2 = 1 - \frac{1}{2!}x^4 + \frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \cdots$$

$$\int_{-1}^{1} \cos x^2 \, dx = \int_{-1}^{1} \left[1 - \frac{1}{2!}x^4 + \frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \cdots\right] dx$$

$$= x - \frac{1}{5(2!)}x^5 + \frac{x^9}{9(4!)} - \frac{x^{13}}{13(6!)} + \cdots \Big|_{-1}^{1}$$

with $n=8$

$$\int_{-1}^{1} \cos x^2 \, dx = x - \frac{1}{5(2!)}x^5 + \frac{x^9}{9(4!)} \Big|_{-1}^{1} = \frac{977}{540} \simeq 1.8$$

H.W: Use a Tayler polynomial with $n=5$ to approximate $\int_{-1}^{1} \frac{\sin x}{x} \, dx$

Ex:
(2) Test for convergence or div.

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \cdots$$

$$= \left(\frac{1}{2} + \frac{1}{2^3} + \cdots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \quad , \text{where}$$

$$a_n = \frac{1}{2^{2n+1}} \quad , \quad b_n = \frac{1}{3^{2n}}$$

$\sum \frac{1}{2^{2n+1}}$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{2n+1}}{2^{2n+3}} = \frac{1}{2^2} = \frac{1}{4} < 1$ Conv.

$\sum \frac{1}{2^{2n}}$, $\lim \frac{b_{n+1}}{b_n} = \frac{1}{3^2} < 1$ Conv.

$\sum (a_n + b_n)$ conv, since, the sum of two conv

H.W Test for Conv. or div.

① $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$ إي احتيا ، يحلو ان نتغير الحل

② $\frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\cdots$

Ex: Examine the convergence of the following series.

(1) $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$, $\sum_{n=1} a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$, (ii) a_n decreasing seq. $a_{n+1} < a_n \ \forall n$

$\Rightarrow$ conv. «conditionally»

(2) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

(3) $\frac{1}{\sqrt{2}+1}-\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{4}+1}-\frac{1}{\sqrt{5}+1}+\cdots$

Ex: For what values of x the following series convergent

(i) $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

(ii) $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

(iii) $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$

~~(iv) $x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\cdots$~~

(iv) $x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\cdots = \sum_{n=1}(-1)^{n-1}\frac{x^n}{n^2}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right| = \lim_{n\to\infty}\left|x\,\frac{n^2}{(n+1)^2}\right|$

$= |x|\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = |x|\cdot 1$ (conv. absol.)

Conv. if $|x|<1 \Rightarrow -1<x<1$

If $x=1$, the series becomes

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$$

$\sum a_n=\sum (-1)^{n-1}\frac{1}{n^2}$ conv. conditionally?

If $x=-1$

$$-1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-\cdots=-\sum\frac{1}{n^2}$$ which is conv.?

the interval of conv. $-1<x\leq 1$.

Fourier Series

periodic functions If the value of each ordinate $f(t)$ repeats itself at equal intervals, then $f(t)$ is said to be a periodic f^n.

If $f(t)=f(t+T)=f(t+2T)=\cdots$ Then T is called the period of the function $f(t)$.

for example

the period of $\sin x$, $\cos x$, $\sec x$, and $\csc x$ is 2π.

The period of $\tan x$ and $\cot x$ is π.

$\sin x=\sin(x+2\pi)=\sin(x+4\pi)=\cdots$

$\sin x$ is periodic function with the period 2π

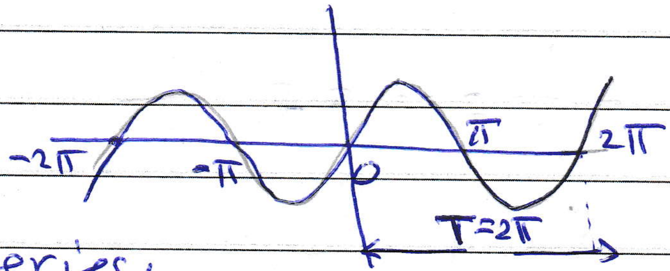
$\sin 5x=\sin(5x+2\pi)=\sin 5\left(x+\frac{2\pi}{5}\right)$, period $=\frac{2\pi}{5}$

$\cos 3x=\cos(3x+2\pi)=\cos 3\left(x+\frac{2\pi}{3}\right)$, period $=\frac{2\pi}{3}$

$$\cos\frac{2n\pi x}{K} = \cos\left(\frac{2n\pi x}{K}+2\pi\right) = \cos\frac{2n\pi}{K}\left(x+\frac{2\pi K}{2n\pi}\right)$$

$$= \cos\frac{2n\pi}{K}\left(x+\frac{K}{n}\right), \text{ period} = \frac{K}{n}$$

$$\tan 2x = \tan(2x+\pi) = \tan 2\left(x+\frac{\pi}{2}\right), \text{ period} = \frac{\pi}{2}$$



Fourier Series:

The following integrals are useful in Fourier series

$$\int_0^{2\pi} \sin nx \, dx = 0 \quad ; \quad \int_0^{2\pi} \cos nx \, dx = 0$$

$$\int_0^{2\pi} \sin^2 nx \, dx = \pi \quad ; \quad \int_0^{2\pi} \cos^2 nx \, dx = \pi$$

$$\int_0^{2\pi} \sin nx \cdot \sin mx \, dx = 0 \,; \quad \int_0^{2\pi} \cos nx \cos mx \, dx = 0$$

$$\int_0^{2\pi} \sin nx \cos mx \, dx = 0 \,; \quad \int_0^{2\pi} \sin nx \cos nx \, dx = 0$$

$$\sin n\pi = 0, \quad \cos n\pi = (-1)^n, \text{ where } n \in \mathbb{Z}$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + a_n\cos nx$$
$$+ \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots + b_n\sin nx + \cdots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

where $a_0, a_1, a_2, \ldots, a_n, \ldots; b_0, b_1, b_2, \ldots, b_n, \ldots$ are constants.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \;;\; a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \;;$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Ex: 1 Find the Fourier Series representing

$f(x) = x \;,\; 0 < x < 2\pi$ and sketch its graph from $x = -4\pi$ to $x = 4\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\frac{x^2}{2}\Big|_0^{2\pi} = \frac{1}{2\pi}4\pi^2 = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1\left(-\frac{\cos n\pi}{n^2}\right)\right]\Big|_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{n^2\pi}(1-1) = 0 \quad , \cos 2n\pi = 1 \,,\, n \in \mathbb{Z}.$$

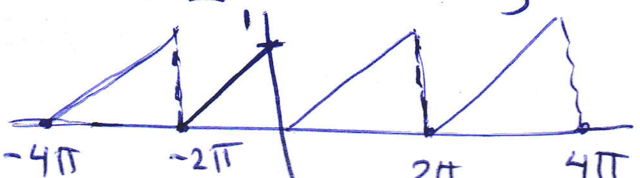
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}$$

$$f(x) = \pi + \sum \left(0\cos nx + \left(-\frac{2}{\pi}\right)\sin nx\right)$$

$$= \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$, but $x = -4\pi$ to $x = 4\pi$

$$f(x) = x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$



Ex (2) Expand in a Fourier series, the function $f(x) = x + x^2$ for $-\pi < x < \pi$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n\pi\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos n\pi\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (2x+1)\frac{-\cos nx}{n^2} + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2} - (-2\pi+1)\frac{\cos(-n\pi)}{n^2}\right] = \frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - (2x+1)\left(-\frac{\sin nx}{n^2}\right) + 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-2\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n$$

$$x + x^2 = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots + b_1\sin x + b_2\sin 2x + \cdots$$

$$= \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$-2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right]$$

Fourier series for discontinuous functions

Let the f^n: $f(x)$ defined by

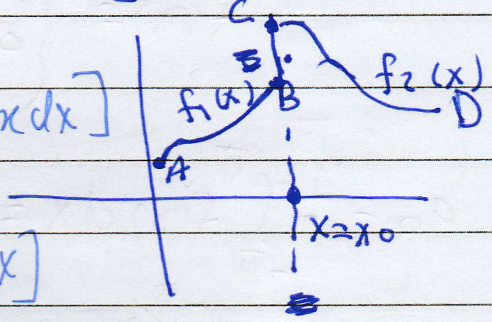
$$f(x) = f_1(x) \quad c < x < x_0$$
$$= f_2(x) \quad x_0 < x < c+2\pi$$

Where x_0 is the point of discont. in the interval $(c, c+2\pi)$.

In such cases also, we obtain the fourier series for $f(x)$ in the usual way. The values a_0, a_n and b_n are defined by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\,dx + \int_{x_0}^{c+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\cos nx\,dx\right]$$



$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\sin nx\,dx\right]$$

If $x = x_0$ is the point of finite discont., then the sum of the Fourier series

$$= \frac{1}{2}\left[\lim_{h\to 0} f(x_0 - h) + \lim_{h\to 0} f(x_0 + h)\right]$$

Ex: 3 Find the fourier series to represent the f^n $f(x)$, given by $f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$;

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_0^{\pi} k\,dx\right] = \frac{1}{\pi}\left[\left[-kx\right]_{-\pi}^{0} + \left[kx\right]_0^{\pi}\right]$$
$$= \frac{1}{\pi}\left[-k\pi + (+k\pi)\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cos nx\,dx + \int_0^{\pi} k\cos nx\,dx\right]$$
$$= \frac{1}{\pi}k\left[-\left.\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left.\frac{\sin nx}{n}\right|_0^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx\,dx + \int_0^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}k\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right] = \frac{k}{\pi}\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$$

if n even $b_n = 0$

if n odd $b_n = \frac{4k}{n\pi}$

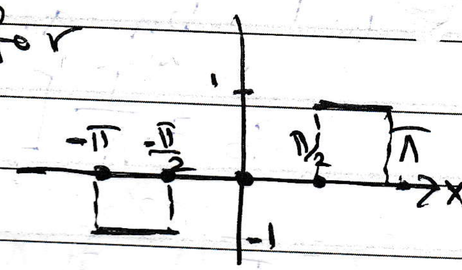
$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \sum b_n \sin nx = b_1 \sin x + b_2 \sin 2x + \cdots$$

$$f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \cdots]$$

Ex (4) Find the fourier series for

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 0\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\,dx$$

$$= \frac{1}{\pi}(-x)\Big|_{-\pi}^{-\pi/2} + \frac{1}{\pi}x\Big|_{\pi/2}^{\pi} = \frac{1}{\pi}\left[\left(\frac{\pi}{2} - \pi\right) + \frac{1}{\pi}\left[\pi - \frac{\pi}{2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 0 \cos nx\,dx$$

$$+ \frac{1}{\pi}\int_{\pi/2}^{\pi} 1 \cos nx\,dx = -\frac{1}{\pi}\left[\frac{\sin nx}{n}\right|_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{\pi}\left[-\frac{\sin \frac{n\pi}{2}}{n}\right] + \frac{1}{\pi}\left[-\frac{\sin \frac{n\pi}{2}}{n}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\right|_{-\pi}^{-\pi/2} - \frac{1}{\pi}\frac{\cos nx}{n}\Big|_{\pi/2}^{\pi} = \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$- \frac{1}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$b_1 = \frac{2}{\pi}$, $b_2 = -\frac{2}{\pi}$, $b_3 = \frac{2}{3\pi}$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

$$= \frac{1}{\pi}\left[2\sin x + 2\sin 2x + \frac{2}{3}\sin 3x + \cdots\right]$$

Ex: In ex(3) show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$f(x) = \frac{4K}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

putting $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = K = \frac{4K}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{3} + \frac{1}{5}\sin\frac{5\pi}{2} + \cdots\right]$$

بعوض K

$$1 = \frac{4}{\pi}\left[1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \cdots\right]$$

$$= \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

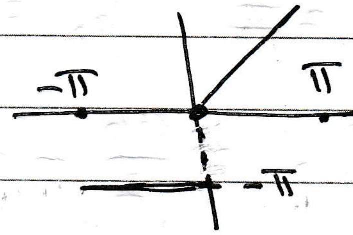
Discontinuous Functions: At a point of discont., Fourier series gives the value of $f(x)$ as the arithmetic mean of the left and right limits.

At the point of discont., $x = c$

at $x = c$, $f(x) = \frac{1}{2}\left[f(c-0) + f(c+0)\right]$

EX (5) Find the Fourier series for

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$



Deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\pi\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\pi\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi n^2}(\cos n\pi - 1)$$, $a_1 = \frac{-2}{\pi}$, $a_2 = 0$, $a_3 = -\frac{2}{9\pi}$, ...

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\pi\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{n}(1 - 2\cos n\pi)$$; $b_1 = 3$, $b_2 = -\frac{1}{2}$, ...

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots \qquad (1)$$

Putting $x = 0$ in (1), we get

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \qquad (2)$$

$f(x)$ is discont. at $x = 0$, but

$f(0-0) = -\pi$, and $f(0+0) = 0$

$$f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\frac{\pi}{2}$$

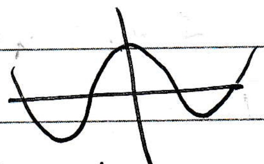
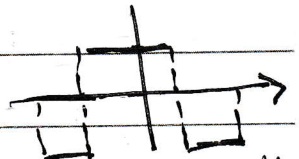
from (2) $$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Even and odd functions

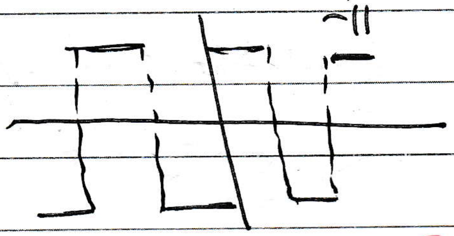
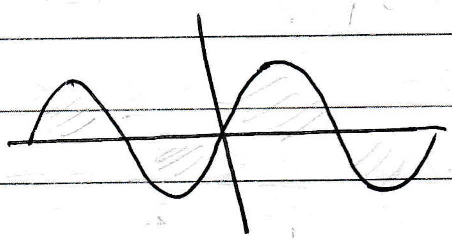
(a) even f^n.
A f^n $f(x)$ is said to be even (or symmetric) f^n
if $f(x) = f(-x)$, the graph sym. w.r.to y-axis
and the area under such a curve from $-\pi$ to π
is double the area from 0 to π

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$$

(b) odd f^n, $f(x)$ is said to be odd (or skew sym.) f^n
if $f(-x) = -f(x)$
the area under the curve from $-\pi$ to π is zero

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$

Expansion of an even f^n.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

As $f(x)$ and $\cos nx$ are both even f^ns, therefore
the product of $f(x)\cos nx$ is also an even f^ns

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

As $\sin nx$ is an odd f^n so $f(x)\cdot\sin nx$ is also an odd f^n.
we need not to calculate b_n.
$\Rightarrow$ the series of the even f^n will contain only cosine terms.

Expansion of an odd f^n

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \quad [f(x)\cos nx \text{ is odd } f^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$[f(x)\cdot\sin nx \text{ is even } f^n]$$

The series of the odd f^n will contain only sine terms.

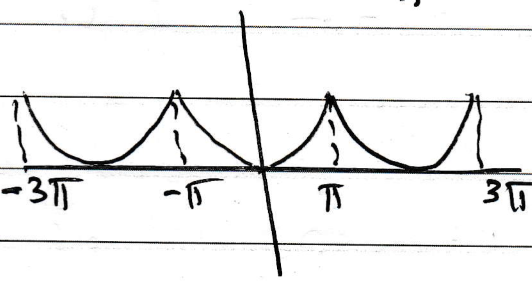
Ex (6) Find the fourier series expansion of the periodic function of period 2π.

$f(x) = x^2, \; -\pi \le x \le \pi$. Hence find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

$f(x) = x^2$ is even f^n. $\Rightarrow b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{4(-1)^n}{n^2}$$



$$f(x) = x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right]$$

putting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$