The Legendre equation is

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

where α is a constant.

Section 5.3, Exercise 22

Determine two linearly independent solutions in powers of x for $|x| < 1$.

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and substitute this into Legendre's equation.

$$\begin{aligned}
0 &= (1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n(n+1)-\alpha(\alpha+1))a_n\right]x^n
\end{aligned}$$

We obtain the recurrence relation:

$$a_{n+2} = \frac{(n(n+1)-\alpha(\alpha+1))a_n}{(n+2)(n+1)}.$$

If we let $a_0 = 1$ and $a_1 = 0$ then we can calculate the following coefficients.

$$\begin{aligned}
a_2 &= -\frac{\alpha(\alpha+1)}{2!} \\
a_4 &= \frac{(2(3)-\alpha(\alpha+1))a_2}{(4)(3)} \\
&= \frac{\alpha(\alpha+1)[\alpha^2+\alpha-6]}{4!} \\
&= \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!} \\
a_6 &= \frac{(4(5)-\alpha(\alpha+1))a_4}{(6)(5)} \\
&= -\frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)(\alpha^2+\alpha-20)}{6!} \\
&= -\frac{\alpha(\alpha-2)(\alpha-4)(\alpha+1)(\alpha+3)(\alpha+5)}{6!}
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
a_{2m} &= (-1)^m \frac{\alpha \cdots (\alpha - 2m + 2)(\alpha + 1) \cdots (\alpha + 2m - 1)}{(2m)!}
\end{aligned}
$$

Hence we have one solution

$$
\begin{aligned}
y_1(x) &= 1 - \frac{\alpha(\alpha+1)}{2!} x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!} x^4 \\
&\quad + \sum_{m=3}^{\infty} \frac{\alpha \cdots (\alpha - 2m + 2)(\alpha + 1) \cdots (\alpha + 2m - 1)}{(2m)!} x^{2m}.
\end{aligned}
$$

Using the Ratio Test for absolute convergence we see that

$$
\begin{aligned}
\lim_{m \to \infty} \left| \frac{a_{2(m+1)} x^{2(m+1)}}{a_{2m} x^{2m}} \right| &= \lim_{m \to \infty} \left| \frac{\frac{\alpha \cdots (\alpha - 2m)(\alpha+1) \cdots (\alpha + 2m + 1)}{(2m+2)!} x^{2m+2}}{\frac{\alpha \cdots (\alpha - 2m + 2)(\alpha + 1) \cdots (\alpha + 2m - 1)}{(2m)!} x^{2m}} \right| \\
&= \lim_{m \to \infty} \left| \frac{(\alpha - 2m)(\alpha + 2m + 1)}{(2m+2)(2m+1)} x^2 \right| \\
&= |x|^2.
\end{aligned}
$$

Thus the power series converges for $|x| < 1$.

If we let $a_0 = 0$ and $a_1 = 1$ then we can calculate the following coefficients.

$$
\begin{aligned}
a_3 &= \frac{1(2) - \alpha(\alpha+1)}{3!} \\
&= -\frac{(\alpha-1)(\alpha+2)}{3!} \\
a_5 &= \frac{(3(4) - \alpha(\alpha+1))a_3}{(5)(4)} \\
&= \frac{(\alpha-1)(\alpha+2)(\alpha^2+\alpha-12)}{5!} \\
&= \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!} \\
a_7 &= \frac{(5(6) - \alpha(\alpha+1))a_5}{(7)(6)} \\
&= -\frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)(\alpha^2+\alpha-30)}{7!} \\
&= -\frac{(\alpha-1)(\alpha-3)(\alpha-5)(\alpha+2)(\alpha+4)(\alpha+6)}{7!} \\
&\vdots \\
a_{2m+1} &= (-1)^m \frac{(\alpha-1) \cdots (\alpha - 2m + 1)(\alpha + 2) \cdots (\alpha + 2m)}{(2m+1)!}
\end{aligned}
$$

Hence we have another solution

$$\begin{aligned} y_2(x) &= x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 \\ &\quad + \sum_{m=3}^{\infty}(-1)^m \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}. \end{aligned}$$

Using the Ratio Test for absolute convergence we see that

$$\begin{aligned} \lim_{m\to\infty}\left|\frac{a_{2(m+1)+1}x^{2(m+1)+1}}{a_{2m+1}x^{2m+1}}\right| &= \lim_{m\to\infty}\left|\frac{\frac{(\alpha-1)\cdots(\alpha-2m-1)(\alpha+2)\cdots(\alpha+2m+2)}{(2m+3)!}x^{2m+3}}{\frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}}\right| \\ &= \lim_{m\to\infty}\left|\frac{(\alpha-2m-1)(\alpha+2m+2)}{(2m+3)(2m+2)}x^2\right| \\ &= |x|^2. \end{aligned}$$

Thus the power series converges for $|x| < 1$.

We may verify the solutions are linearly independent using the Wronskian.

$$W(y_1, y_2)(0) = y_1(0)y_2'(0) - y_1'(0)y_2(0) = 1 \neq 0$$

Section 5.3, Exercise 23

Show that if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x. Find the polynomials corresponding to $\alpha = 0$, 2, and 4. Show that if α is a positive odd integer $2n+1$, the series solution y_2 reduces to a polynomial of degree $2n+1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1$, 3, and 5.

If $\alpha = 0$ then $y_1(x) = 1$ which is a polynomial of degree zero. If $\alpha = 2n$ for some $n \in \mathbb{N}$ then according to the recurrence relation $a_{2n+2k} = 0$ for all $k \in \mathbb{N}$ which implies that

$$\begin{aligned} y_1(x) &= 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 \\ &\quad + \sum_{m=3}^{2n}\frac{\alpha\cdots(\alpha-2m+2)(\alpha+1)\cdots(\alpha+2m-1)}{(2m)!}x^{2m}, \end{aligned}$$

a polynomial of degree $2n$ containing only even powers of x.

$$\begin{aligned} \alpha = 0&: \quad y(x) = 1 \\ \alpha = 2&: \quad y(x) = 1 - 3x^2 \\ \alpha = 4&: \quad y(x) = 1 - 10x^2 + \frac{35}{3}x^4 \end{aligned}$$

If $\alpha = 1$ then $y_2(x) = x$ which is a polynomial of degree 1 containing only odd powers of x. If $\alpha = 2n+1$ for some $n \in \mathbb{N}$ then according to the recurrence relation $a_{2n+1+2k} = 0$ for all $k \in \mathbb{N}$ which implies that

$$\begin{aligned} y_2(x) &= x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 \\ &\quad + \sum_{m=3}^{2n+1}(-1)^m \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}, \end{aligned}$$

a polynomial of degree $2n+1$ containing only odd powers of x.

$$\begin{aligned} \alpha = 1&: \quad y(x) = x \\ \alpha = 3&: \quad y(x) = x - \frac{5}{3}x^3 \\ \alpha = 5&: \quad y(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5 \end{aligned}$$

Section 5.3, Exercise 24

The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.

(a) Using the results of Problem 23, find the Legendre polynomials $P_0(x), \ldots, P_5(x)$.

We can easily see that $P_0(x) = 1$ and $P_1(x) = x$. Since

$$\begin{aligned} P_2(x) &= a(1-3x^2) \\ P_2(1) &= a(1-3) = 1 \end{aligned}$$

then $a = -1/2$ and we have

$$P_2(x) = -\frac{1}{2}(1-3x^2).$$

In the same way we see that

$$\begin{aligned} P_3(x) &= a\left(x - \frac{5}{3}x^3\right) \\ P_3(1) &= a\left(1 - \frac{5}{3}\right) = 1 \\ a &= -\frac{3}{2} \\ P_4(x) &= a\left(1 - 10x^2 + \frac{35}{3}x^4\right) \\ P_4(1) &= a\left(1 - 10 + \frac{35}{3}\right) = 1 \end{aligned}$$

$$
\begin{aligned}
a &= \frac{3}{8} \\
P_5(x) &= a\left(x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right) \\
P_5(1) &= a\left(1 - \frac{14}{3} + \frac{21}{5}\right) = 1 \\
a &= \frac{15}{8}.
\end{aligned}
$$

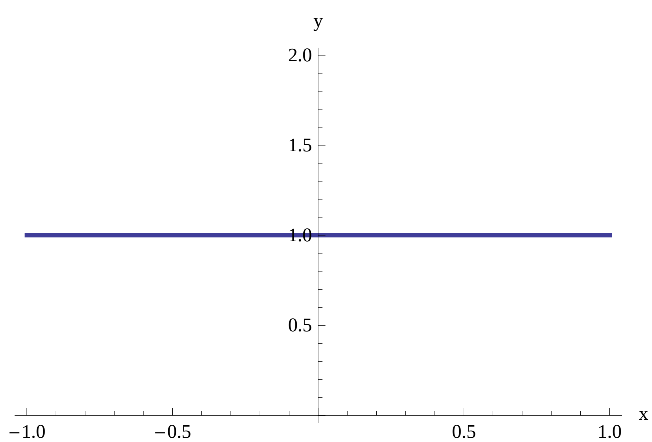
To summarize

Legendre Polynomials

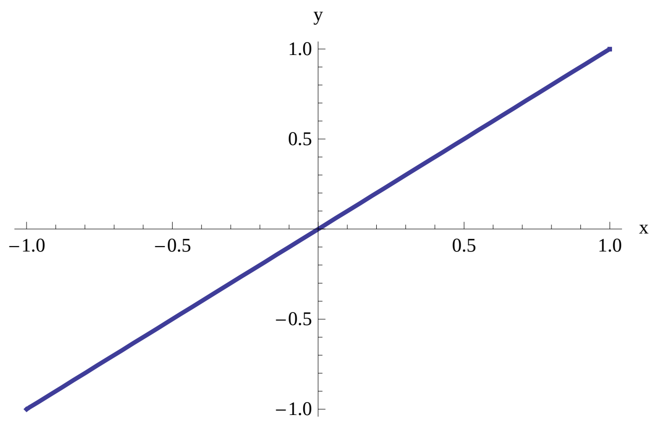
$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x)
\end{aligned}
$$

(b) Plot the graphs of $P_0(x)$, …, $P_5(x)$ for $-1 \leq x \leq 1$.

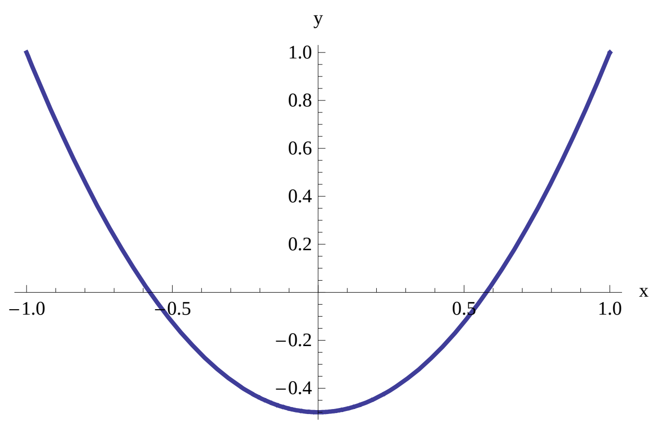
$P_0(x) = 1$



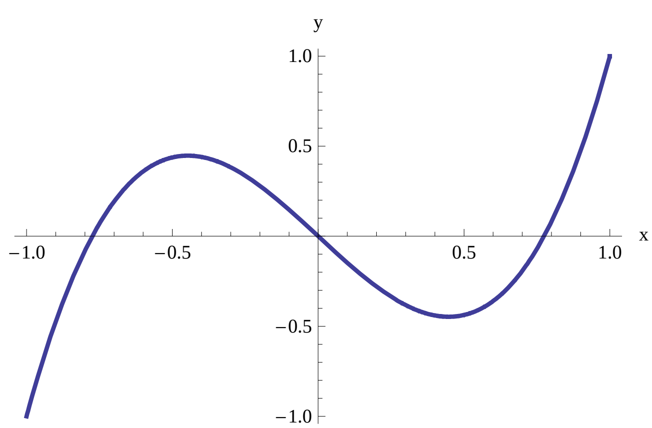
$$P_1(x) = x$$



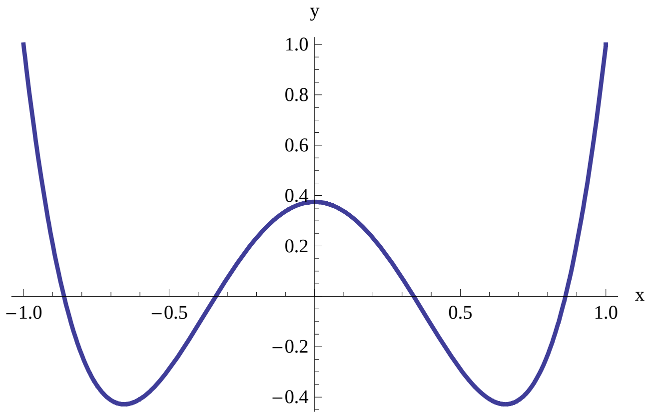
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$



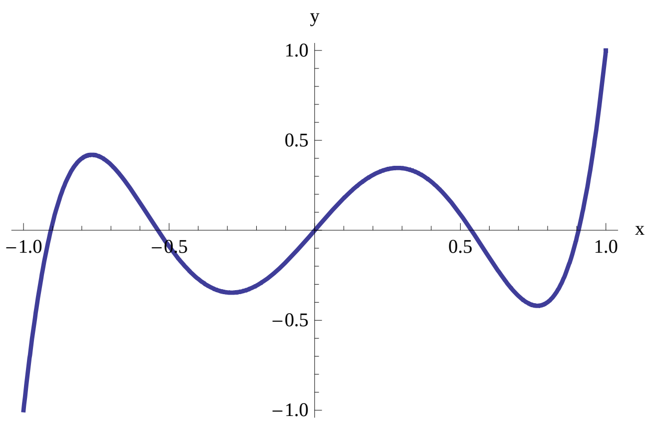
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$



$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$



$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$



(c) Find the zeros of $P_0(x), \ldots, P_5(x)$.

$P_0(x) = 1$ has no zeros.

$P_1(x) = x$ has a zero at $x = 0$.

$P_2(x) = \frac{1}{2}(3x^2 - 1)$ has zeros at $x = \pm\dfrac{1}{\sqrt{3}}$.

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$ has zeros at $x = 0$ and at $x = \pm\sqrt{\dfrac{3}{5}}$.

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ has zeros at $x = \pm\sqrt{\dfrac{15 \pm \sqrt{30}}{35}}$.

$\frac{1}{8}(63x^5 - 70x^3 + 15x)$ has zeros at $x = 0$ and at $x = \pm\sqrt{\dfrac{35 \pm 2\sqrt{70}}{63}}$.

Section 5.3, Exercise 25

It can be shown that the general formula for $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k},$$

where $[n/2]$ denotes the greatest integer less than or equal to $n/2$. By observing the form of $P_n(x)$ for n even and n odd, show that $P_n(-1) = (-1)^n$.

By exercise 23 we know $P_n(x)$ is a polynomial of even (odd) degree when n is even (odd) and contains only even (odd) powers of x. Thus $P_n(x)$ is an even (odd) function when n is even (odd). By exercise 24, $P_n(1) = 1$, thus when n is even $P_n(-1) = 1 = (-1)^n$ and when n is odd $P_n(-1) = -P_n(1) = -1 = (-1)^n$.

Section 5.3, Exercise 26

The Legendre polynomials play an important role in mathematical physics. For example in solving Laplace's equation (the potential equation) in spherical coordinates, we encounter the equation

$$\frac{d^2F(\varphi)}{d\varphi^2} + \cot\varphi \frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) = 0, \qquad 0 < \varphi < \pi,$$

where n is a positive integer. Show that the change of variable $x = \cos\varphi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\arccos x)$.

Using the chain rule for derivatives we see that

$$\frac{dF(\varphi)}{d\varphi} = \frac{dF(\varphi)}{dx}\frac{dx}{d\varphi} = \frac{dF(\arccos x)}{dx}(-\sin\varphi) = \frac{df}{dx}(-\sin\varphi) = -\sin\varphi y'.$$

Differentiating once more we have

$$\frac{d^2F(\varphi)}{d\varphi^2} = \frac{d}{d\varphi}[-\sin\varphi y'] = -\cos\varphi y' + \sin^2\varphi y''.$$

Substituting into the equation above we have

$$\begin{aligned} 0 &= 0 - \cos\varphi y' + \sin^2\varphi y'' + \cot\varphi(-\sin\varphi y') + n(n+1)y \\ &= \sin^2\varphi y'' - 2\cos\varphi y' + n(n+1)y \\ &= (1-\cos^2\varphi)y'' - 2\cos\varphi y' + n(n+1)y \\ &= (1-x^2)y'' - 2xy' + n(n+1)y. \end{aligned}$$

Section 5.3, Exercise 27

Show that for $n = 0, 1, 2, 3$, the corresponding Legendre polynomial is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

This formula, known as Rodrigues' formula, is true for all positive integers n.

For the case $n = 0$

$$P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0} (x^2 - 1)^0 = 1.$$

For the case $n = 1$

$$P_1(x) = \frac{1}{2^1 1!} \frac{d}{dx} (x^2 - 1) = \frac{1}{2}(2x) = x.$$

For the case $n = 2$

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2 - 1)^2 = \frac{1}{8} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1) = \frac{1}{8} \frac{d}{dx} (4x^3 - 4x) = \frac{1}{8}(12x^2 - 4) \\ &= \frac{3}{2}x^2 - \frac{1}{2}. \end{aligned}$$

For the case $n = 3$

$$\begin{aligned} P_3(x) &= \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2 - 1)^3 = \frac{1}{48} \frac{d^3}{dx^3} (x^6 - 3x^4 + 3x^2 - 1) = \frac{1}{48} \frac{d^2}{dx^2} (6x^5 - 12x^3 + 6x) \\ &= \frac{1}{48} \frac{d}{dx} (30x^4 - 36x^2 + 6) = \frac{1}{48}(120x^3 - 72x) \\ &= \frac{5}{2}x^3 - \frac{3}{2}x. \end{aligned}$$

Section 5.3, Exercise 28

Show that the Legendre equation can also be written as

$$[(1 - x^2)y']' = -\alpha(\alpha + 1)y.$$

Then it follows that

$$[(1 - x^2)P_n'(x)]' = -n(n + 1)P_n(x) \quad \text{and} \quad [(1 - x^2)P_m'(x)]' = -m(m + 1)P_m(x).$$

By multiplying the first equation by $P_m(x)$ and the second equation by $P_n(x)$, integrating by parts, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \quad \text{if } n \neq m.$$

This property of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be show that the value of the preceding integral is $2/(2n + 1)$.

Starting with the Legendre equation

$$\begin{aligned} 0 &= (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y \\ &= (1-x^2)(y')' + (1-x^2)'y' + \alpha(\alpha+1)y \\ -\alpha(\alpha+1)y &= [(1-x^2)y']' \quad \text{(product rule)}. \end{aligned}$$

Thus when $y(x) = P_n(x)$ we have

$$[(1-x^2)P_n'(x)]' = -n(n+1)P_n(x).$$

Suppose we multiply this equation by $P_m(x)$ where $m \neq n$, then

$$\begin{aligned} [(1-x^2)P_n'(x)]'P_m(x) &= -n(n+1)P_n(x)P_m(x) \\ \int_{-1}^{1} [(1-x^2)P_n'(x)]'P_m(x)\,dx &= -n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx \end{aligned}$$

We can apply integration by parts to the integral on the left-hand side of this equation.

$$\begin{aligned} u &= P_m(x) & v &= (1-x^2)P_n'(x) \\ du &= P_m'(x) & dv &= [(1-x^2)P_n'(x)]'\,dx \end{aligned}$$

Thus

$$\begin{aligned} \int_{-1}^{1} [(1-x^2)P_n'(x)]'P_m(x)\,dx &= P_m(x)[(1-x^2)P_n'(x)]\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx \\ &= -\int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx. \end{aligned}$$

Similarly when $y(x) = P_m(x)$ we have

$$[(1-x^2)P_m'(x)]' = -m(m+1)P_m(x).$$

Suppose we multiply this equation by $P_n(x)$, then

$$\begin{aligned} [(1-x^2)P_m'(x)]'P_n(x) &= -m(m+1)P_m(x)P_n(x) \\ \int_{-1}^{1} [(1-x^2)P_m'(x)]'P_n(x)\,dx &= -m(m+1)\int_{-1}^{1} P_m(x)P_n(x)\,dx \end{aligned}$$

We can apply integration by parts to the integral on the left-hand side of this equation.

$$\begin{aligned} u &= P_n(x) & v &= (1-x^2)P_m'(x) \\ du &= P_n'(x) & dv &= [(1-x^2)P_m'(x)]'\,dx \end{aligned}$$

Thus

$$\begin{aligned} \int_{-1}^{1} [(1-x^2)P_m'(x)]'P_n(x)\,dx &= P_n(x)[(1-x^2)P_m'(x)]\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx \\ &= -\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx. \end{aligned}$$

Combining the equations we see that

$$\begin{aligned}
\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx &= \int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx \\
m(m+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx &= n(n+1)\int_{-1}^{1}P_n(x)P_m(x)\,dx \\
[m(m+1)-n(n+1)]\int_{-1}^{1}P_m(x)P_n(x)\,dx &= 0.
\end{aligned}$$

Thus either

$$\int_{-1}^{1}P_m(x)P_n(x)\,dx = 0$$

which is the orthogonality property, or

$$0 = [m(m+1)-n(n+1)] = (m-n)(m+n+1)$$

Since m, $n \in \mathbb{N}$ and we have assumed $m \neq n$ then this equation is not satisfied.

Section 5.3, Exercise 29

Given a polynomial f of degree n, it is possible to express f as a linear combination of P_0, P_1, P_2, $\ldots$, P_n:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x).$$

Using the result of Problem 28, show that

$$a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)\,dx.$$

Suppose

$$f(x) = \sum_{k=0}^{n} a_k P_k(x).$$

then

$$\begin{aligned}
f(x)P_m(x) &= \sum_{k=0}^{n} a_k P_k(x)P_m(x) \\
\int_{-1}^{1} f(x)P_m(x)\,dx &= \sum_{k=0}^{n} a_k \int_{-1}^{1} P_k(x)P_m(x)\,dx \\
&= \frac{2}{2m+1}a_m \\
\frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)\,dx &= a_m
\end{aligned}$$