Curvilinear Coordinates

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The governing equations were derived using the most basic coordinate system, i.e, Cartesian coordinates:

$$\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl}\,\mathbf{f} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\text{Laplacian} = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Example: incompressible flow equations

$$\nabla \cdot \mathbf{V} = 0$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla\left(p + \gamma z\right) + \mu \nabla^2 \mathbf{V}$$

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}\right) = -\nabla\left(p + \gamma z\right) + \mu \nabla^2 \mathbf{V}$$

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + \frac{1}{2}\nabla\left(\mathbf{V}\cdot\mathbf{V}\right) - \mathbf{V}\times\boldsymbol{\omega}\right] = -\nabla\left(p + \gamma z\right) + \mu\left[\nabla\left(\nabla\cdot\mathbf{V}\right) - \nabla\times\boldsymbol{\omega}\right]$$

$$\boldsymbol{\omega} = \nabla \times \mathbf{V}$$

$\nabla\left(\nabla\cdot\mathbf{V}\right) = 0$ in the above equation, but retained to keep the complete vector identity for $\nabla^2\mathbf{V}$ in equation.

However, once the equations are expressed in vector invariant form (as above) they can be transformed into any convenient coordinate system through the use of appropriate definitions for the $\nabla, \nabla\cdot, \nabla\times$, and ∇^2. Frequently, alternative coordinate systems are desirable which either exploit certain features of the flow at hand or facilitate numerical procedures. The most general coordinate system for fluid flow problems are nonorthogonal curvilinear coordinates. A special case of these are orthogonal curvilinear coordinates. Here we shall derive the appropriate relations for the latter using vector technique. It should be recognized that the derivation can also be accomplished using tensor analysis

1. Orthogonal curvilinear coordinate systems

Suppose that the Cartesian coordinates (x, y, z) are expressed in terms of the new coordinates (x_1, x_2, x_3) by the equations

$$x = x(x_1, x_2, x_3)$$
$$y = y(x_1, x_2, x_3)$$
$$z = z(x_1, x_2, x_3)$$

where it is assumed that the correspondence is unique and that the inverse mapping exists.

For example, circular cylindrical coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

i.e., at any point P, x_1 curve is a straight line, x_2 curve is a circle, and the x_3 curve is a straight line.

The position vector of a point P in space is

$$\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$\mathbf{R} = (r\cos\theta)\hat{\mathbf{i}} + (r\sin\theta)\hat{\mathbf{j}} + (z)\hat{\mathbf{k}}$ for cylindrical coordinates

By definition a vector tangent to the x_1 curve is given by:

$\mathbf{R}_{x_1} = x_{x_1}\hat{\mathbf{i}} + y_{x_1}\hat{\mathbf{j}} + z_{x_1}\hat{\mathbf{k}}$ (Subscript denotes partial differentiation)

So that the unit vectors tangent to the x_i curve are

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{R}_{x_1}}{h_1}, \qquad \hat{\mathbf{e}}_2 = \frac{\mathbf{R}_{x_2}}{h_2}, \qquad \hat{\mathbf{e}}_3 = \frac{\mathbf{R}_{x_3}}{h_3}$$

Where $h_i = |\mathbf{R}_{x_1}|$ are called the metric coefficients or scale factors

$h_r = 1, \qquad h_\theta = r, \qquad h_z = 1$ for cylindrical coordinates

The arc length along a curve in any direction is given by

$$ds^2 = d\mathbf{R}\cdot d\mathbf{R} = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2$$

Since $d\mathbf{R} = \mathbf{R}_{x_i} dx_i = h_i dx_i \hat{\mathbf{e}}_i$ and $\mathbf{R}_{x_i} = h_i \hat{\mathbf{e}}_i$

and since the x_i are orthogonal: $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

An element of volume is given by the triple product
$d\forall = \left(h_1 dx_1 \hat{\mathbf{e}}_1 \times h_2 dx_2 \hat{\mathbf{e}}_2\right) \cdot h_3 dx_3 \hat{\mathbf{e}}_3 = h_1 h_2 h_3 dx_1 dx_2 dx_3$
Where since the x_i are orthogonal $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3$

Finally, on the surface $x_1 =$ constant, the vector element of surface area is given by
$d\mathbf{s}_1 = h_2 dx_2 \hat{\mathbf{e}}_2 \times h_3 dx_3 \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 h_2 h_3 dx_2 dx_3$
With similar results for x_2 and $x_3 =$ constant
$d\mathbf{s}_2 = \hat{\mathbf{e}}_2 h_3 h_1 dx_3 dx_1$
$d\mathbf{s}_3 = \hat{\mathbf{e}}_3 h_1 h_2 dx_1 dx_2$

2. Differential operators in orthogonal curvilinear coordinate systems

With the above in hand, we now proceed to obtain the desired vector operators

2.1 Gradient $\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial f}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial f}{\partial x_3}\hat{\mathbf{e}}_3$

By definition: $df = \nabla f \cdot d\mathbf{R} = f_{x_i} dx_i$
If we temporarily write $\nabla f = \lambda_1 \hat{\mathbf{e}}_1 + \lambda_2 \hat{\mathbf{e}}_2 + \lambda_3 \hat{\mathbf{e}}_3$
Then by comparison

$$df = f_{x_i} dx_i = \lambda_i h_i dx_i$$

$$\lambda_i = \frac{1}{h_i}\frac{\partial f}{\partial x_i}$$

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3$$

$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial}{\partial z}\hat{\mathbf{e}}_z$ for cylindrical coordinates

Note $\nabla x_i = \frac{\hat{\mathbf{e}}_i}{h_i}$

$$\lambda_i = \frac{1}{h_i}\frac{\partial f}{\partial x_i}$$

So that by definition ($\mathrm{curl}\left(\mathrm{grad}\ f\right) = 0$)

$$\nabla\times\nabla x_i = \nabla\times\frac{\hat{\mathbf{e}}_i}{h_i} = 0$$

Also $\dfrac{\hat{\mathbf{e}}_1}{h_2h_3} = \dfrac{\hat{\mathbf{e}}_2}{h_2}\times\dfrac{\hat{\mathbf{e}}_3}{h_3} = \nabla x_2\times\nabla x_3$

So that by definition ($\nabla\cdot(\nabla f\times\nabla g) = 0$)

$$\nabla\cdot\left(\frac{\hat{\mathbf{e}}_1}{h_2h_3}\right) = \nabla\cdot\left(\frac{\hat{\mathbf{e}}_2}{h_3h_1}\right) = \nabla\cdot\left(\frac{\hat{\mathbf{e}}_3}{h_1h_2}\right) = 0$$

2.2 Divergence $\nabla\cdot\mathbf{F} = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial x_1}(h_2h_3F_1) + \dfrac{\partial}{\partial x_2}(h_3h_1F_2) + \dfrac{\partial}{\partial x_3}(h_1h_2F_3)\right]$

$$\nabla\cdot\mathbf{F} = \nabla\cdot(F_1\hat{\mathbf{e}}_1) + \nabla\cdot(F_2\hat{\mathbf{e}}_2) + \nabla\cdot(F_3\hat{\mathbf{e}}_3)$$

$$\nabla\cdot(F_1\hat{\mathbf{e}}_1) = \nabla\cdot\left[h_2h_3F_1\left(\frac{\hat{\mathbf{e}}_1}{h_2h_3}\right)\right] \text{ using } \nabla\cdot(\varphi\mathbf{u}) = \varphi\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\varphi$$

$$= \frac{\hat{\mathbf{e}}_1}{h_2h_3}\cdot\nabla(h_2h_3F_1) \text{ using } \nabla\cdot\left(\frac{\hat{\mathbf{e}}_1}{h_2h_3}\right) = \nabla\cdot\left(\frac{\hat{\mathbf{e}}_2}{h_3h_1}\right) = \nabla\cdot\left(\frac{\hat{\mathbf{e}}_3}{h_1h_2}\right) = 0$$

$$= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_1}(h_2h_3F_1)$$

Treating the other terms in a similar manner results in

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1) + \frac{\partial}{\partial x_2}(h_3h_1F_2) + \frac{\partial}{\partial x_3}(h_1h_2F_3)\right]$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_1) + \frac{\partial}{\partial\theta}(F_2) + \frac{\partial}{\partial z}(rF_3)\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(rF_1) + \frac{1}{r}\frac{\partial}{\partial\theta}(F_2) + \frac{\partial}{\partial z}(F_3) \text{ for cylindrical coordinates}$$

2.3 Curl $\nabla\times\mathbf{F} = \dfrac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$

$$\nabla\times\mathbf{F} = \nabla\times(F_1\hat{\mathbf{e}}_1) + \nabla\times(F_2\hat{\mathbf{e}}_2) + \nabla\times(F_3\hat{\mathbf{e}}_3)$$

$$\nabla\times(F_1\hat{\mathbf{e}}_1) = \nabla\times\left[(h_1F_1)\left(\frac{\hat{\mathbf{e}}_1}{h_1}\right)\right]$$

$$= -\frac{\hat{\mathbf{e}}_1}{h_1} \times \nabla\left(h_1 F_1\right) \text{ using } \nabla \times (\varphi \mathbf{u}) = \varphi \nabla \times \mathbf{u} + \nabla \varphi \times \mathbf{u} \text{ and } \nabla \times \frac{\hat{\mathbf{e}}_i}{h_i} = 0$$

$$= -\frac{\hat{\mathbf{e}}_1}{h_1} \times \left[\frac{1}{h_1} \frac{\partial\left(h_1 F_1\right)}{\partial x_1} \hat{\mathbf{e}}_1 + \frac{1}{h_2} \frac{\partial\left(h_1 F_1\right)}{\partial x_2} \hat{\mathbf{e}}_2 + \frac{1}{h_3} \frac{\partial\left(h_1 F_1\right)}{\partial x_3} \hat{\mathbf{e}}_3 \right]$$

$$= -\frac{\hat{\mathbf{e}}_3}{h_1 h_2} \frac{\partial}{\partial x_2}\left(h_1 F_1\right) + \frac{\hat{\mathbf{e}}_2}{h_3 h_1} \frac{\partial}{\partial x_3}\left(h_1 F_1\right)$$

$$= \frac{1}{h_1 h_2 h_3} \left[h_2 \hat{\mathbf{e}}_2 \frac{\partial}{\partial x_3} - h_3 \hat{\mathbf{e}}_3 \frac{\partial}{\partial x_2} \right] \left(h_1 F_1\right)$$

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

$$\nabla \times \mathbf{F} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_1 & rF_2 & F_3 \end{vmatrix}$$ for cylindrical coordinates

2.4 Laplacian acting on a scalar $\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial x_1} \right) + \frac{\partial}{\partial x_2}\left(\frac{h_3 h_1}{h_2} \frac{\partial}{\partial x_2} \right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial x_3} \right) \right]$

$$\nabla^2 = \nabla \cdot \nabla$$

$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial x_1} \right) + \frac{\partial}{\partial x_2}\left(\frac{h_3 h_1}{h_2} \frac{\partial}{\partial x_2} \right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial x_3} \right) \right]$$

$$\nabla^2 = \frac{1}{r} \left[\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) + \frac{\partial}{\partial z}\left(r \frac{\partial}{\partial z} \right) \right]$$

$$= \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta}\left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) + \frac{1}{r} \frac{\partial}{\partial z}\left(r \frac{\partial}{\partial z} \right)$$

2.5 Laplacian acting on a vector $\nabla^2 \mathbf{F} = \nabla\left(\nabla \cdot \mathbf{F}\right) - \nabla \times \left(\nabla \times \mathbf{F}\right)$

Using $\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial x_1} \hat{\mathbf{e}}_1 + \frac{1}{h_2} \frac{\partial f}{\partial x_2} \hat{\mathbf{e}}_2 + \frac{1}{h_3} \frac{\partial f}{\partial x_3} \hat{\mathbf{e}}_3$

and $\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1}\left(h_2 h_3 F_1\right) + \frac{\partial}{\partial x_2}\left(h_3 h_1 F_2\right) + \frac{\partial}{\partial x_3}\left(h_1 h_2 F_3\right) \right]$

$$\nabla\left(\nabla \cdot \mathbf{F}\right) =$$

$$\frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_1$$
$$+\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_2$$
$$+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_3$$

Using $\nabla\times\mathbf{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$

$\nabla\times(\nabla\times\mathbf{F})=$

$$=\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)\right]\hat{\mathbf{e}}_1$$
$$+\frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)\right]\hat{\mathbf{e}}_2$$
$$+\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)\right]\hat{\mathbf{e}}_3$$

Combining those two terms gives

$\nabla^2\mathbf{F}=\nabla(\nabla\cdot\mathbf{F})-\nabla\times(\nabla\times\mathbf{F})=$

$$=\frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_1$$
$$-\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)\right]\hat{\mathbf{e}}_1$$
$$+\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_2$$
$$-\frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)\right]\hat{\mathbf{e}}_2$$
$$+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_3$$
$$-\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)\right]\hat{\mathbf{e}}_3$$

For cylindrical coordinates (r,θ,z), $h_1=h_r=1,\ \ h_2=h_\theta=r,\ \ h_3=h_z=1$, and use the definition of Laplacian operator acting on a scalar $\nabla^2 f$

$$
\begin{aligned}
\nabla^2 f &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right)\right]\\
&=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\\
&=\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}
\end{aligned}
$$

$$
\nabla^2\mathbf{F}=a\hat{\mathbf{e}}_r+b\hat{\mathbf{e}}_\theta+c\hat{\mathbf{e}}_z=\left(\nabla^2F_1-\frac{1}{r^2}F_1-\frac{2}{r^2}\frac{\partial F_2}{\partial \theta}\right)\hat{\mathbf{e}}_r+\left(\nabla^2F_2-\frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial \theta}\right)\hat{\mathbf{e}}_\theta+\left(\nabla^2F_3\right)\hat{\mathbf{e}}_z
$$

$$
\begin{aligned}
a &=\\
&=\frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(h_2h_3F_1\right)+\frac{\partial}{\partial x_2}\left(h_3h_1F_2\right)+\frac{\partial}{\partial x_3}\left(h_1h_2F_3\right)\right]\right]\\
&-\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(h_2F_2\right)-\frac{\partial}{\partial x_2}\left(h_1F_1\right)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(h_1F_1\right)-\frac{\partial}{\partial x_1}\left(h_3F_3\right)\right]\right)\right]\\
&=\frac{\partial}{\partial r}\left[\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_1\right)+\frac{\partial}{\partial \theta}\left(F_2\right)+\frac{\partial}{\partial z}\left(rF_3\right)\right]\right]\\
&-\frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_2\right)-\frac{\partial}{\partial \theta}\left(F_1\right)\right]\right)-\frac{\partial}{\partial z}\left(r\left[\frac{\partial}{\partial z}\left(F_1\right)-\frac{\partial}{\partial r}\left(F_3\right)\right]\right)\right]\\
&=\frac{\partial}{\partial r}\left[\frac{1}{r}\left[F_1+r\frac{\partial F_1}{\partial r}+\frac{\partial F_2}{\partial \theta}+r\frac{\partial F_3}{\partial z}\right]\right]\\
&-\frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{1}{r}\left[F_2+r\frac{\partial F_2}{\partial r}-\frac{\partial F_1}{\partial \theta}\right]\right)-\frac{\partial}{\partial z}\left(r\frac{\partial F_1}{\partial z}-r\frac{\partial F_3}{\partial r}\right)\right]\\
&=\frac{\partial}{\partial r}\left[\frac{1}{r}F_1+\frac{\partial F_1}{\partial r}+\frac{1}{r}\frac{\partial F_2}{\partial \theta}+\frac{\partial F_3}{\partial z}\right]\\
&-\frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{1}{r}F_2+\frac{\partial F_2}{\partial r}-\frac{1}{r}\frac{\partial F_1}{\partial \theta}\right)-r\frac{\partial^2F_1}{\partial z^2}+r\frac{\partial^2F_3}{\partial z\partial r}\right]\\
&=\left(\frac{-1}{r^2}F_1\right)+\frac{1}{r}\frac{\partial F_1}{\partial r}+\frac{\partial^2F_1}{\partial r^2}+\left(\frac{-1}{r^2}\frac{\partial F_2}{\partial \theta}\right)+\frac{1}{r}\frac{\partial^2F_2}{\partial r\partial \theta}+\frac{\partial^2F_3}{\partial r\partial z}\\
&-\frac{1}{r}\left[\frac{1}{r}\frac{\partial F_2}{\partial \theta}+\frac{\partial^2F_2}{\partial \theta\partial r}-\frac{1}{r}\frac{\partial^2F_1}{\partial \theta^2}-r\frac{\partial^2F_1}{\partial z^2}+r\frac{\partial^2F_3}{\partial z\partial r}\right]\\
&=\frac{-1}{r^2}F_1+\frac{1}{r}\frac{\partial F_1}{\partial r}+\frac{\partial^2F_1}{\partial r^2}-\frac{1}{r^2}\frac{\partial F_2}{\partial \theta}+\cancel{\frac{1}{r}\frac{\partial^2F_2}{\partial r\partial \theta}}+\cancel{\frac{\partial^2F_3}{\partial r\partial z}}
\end{aligned}
$$

$$+\left[-\frac{1}{r^2}\frac{\partial F_2}{\partial\theta}-\cancel{\frac{1}{r}\frac{\partial^2 F_2}{\partial\theta\partial r}}+\frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2}+\frac{\partial^2 F_1}{\partial z^2}-\cancel{\frac{\partial^2 F_3}{\partial z\partial r}}\right]$$

$$=\frac{-1}{r^2}F_1+\frac{1}{r}\frac{\partial F_1}{\partial r}+\frac{\partial^2 F_1}{\partial r^2}-\frac{2}{r^2}\frac{\partial F_2}{\partial\theta}+\left[\frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2}+\frac{\partial^2 F_1}{\partial z^2}\right]$$

$$=\left(\frac{1}{r}\frac{\partial F_1}{\partial r}+\frac{\partial^2 F_1}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2}+\frac{\partial^2 F_1}{\partial z^2}\right)-\frac{1}{r^2}F_1-\frac{2}{r^2}\frac{\partial F_2}{\partial\theta}$$

$$=\left(\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial F_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2}+\frac{\partial^2 F_1}{\partial z^2}\right)-\frac{1}{r^2}F_1-\frac{2}{r^2}\frac{\partial F_2}{\partial\theta}$$

$$=\nabla^2 F_1-\frac{1}{r^2}F_1-\frac{2}{r^2}\frac{\partial F_2}{\partial\theta}$$

$b=$

$$=\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]$$

$$-\frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_1)+\frac{\partial}{\partial\theta}(F_2)+\frac{\partial}{\partial z}(rF_3)\right]\right]$$

$$-\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\left[\frac{\partial}{\partial\theta}(F_3)-\frac{\partial}{\partial z}(rF_2)\right]\right)-\frac{\partial}{\partial r}\left(\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_2)-\frac{\partial}{\partial\theta}(F_1)\right]\right)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r}\left[F_1+r\frac{\partial F_1}{\partial r}+\frac{\partial F_2}{\partial\theta}+r\frac{\partial F_3}{\partial z}\right]\right]$$

$$-\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial F_3}{\partial\theta}-\frac{\partial F_2}{\partial z}\right)-\frac{\partial}{\partial r}\left(\frac{1}{r}\left[F_2+r\frac{\partial F_2}{\partial r}-\frac{\partial F_1}{\partial\theta}\right]\right)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r}F_1+\frac{\partial F_1}{\partial r}+\frac{1}{r}\frac{\partial F_2}{\partial\theta}+\frac{\partial F_3}{\partial z}\right]$$

$$-\left[\frac{1}{r}\frac{\partial^2 F_3}{\partial z\partial\theta}-\frac{\partial^2 F_2}{\partial z^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}F_2+\frac{\partial F_2}{\partial r}-\frac{1}{r}\frac{\partial F_1}{\partial\theta}\right)\right]$$

$$=\frac{1}{r}\left[\frac{1}{r}\frac{\partial F_1}{\partial\theta}+\frac{\partial^2 F_1}{\partial\theta\partial r}+\frac{1}{r}\frac{\partial^2 F_2}{\partial\theta^2}+\frac{\partial^2 F_3}{\partial\theta\partial z}\right]$$

$$-\left[\frac{1}{r}\frac{\partial^2 F_3}{\partial z\partial\theta}-\frac{\partial^2 F_2}{\partial z^2}-\left(\frac{-F_2}{r^2}+\frac{1}{r}\frac{\partial F_2}{\partial r}+\frac{\partial^2 F_2}{\partial r^2}-\frac{-1}{r^2}\frac{\partial F_1}{\partial\theta}-\frac{1}{r}\frac{\partial^2 F_1}{\partial r\partial\theta}\right)\right]$$

$$=\frac{1}{r^2}\frac{\partial F_1}{\partial\theta}+\cancel{\frac{1}{r}\frac{\partial^2 F_1}{\partial\theta\partial r}}+\frac{1}{r^2}\frac{\partial^2 F_2}{\partial\theta^2}+\cancel{\frac{1}{r}\frac{\partial^2 F_3}{\partial\theta\partial z}}$$

$$+\left[-\cancel{\frac{1}{r}\frac{\partial^2 F_3}{\partial z\partial\theta}}+\frac{\partial^2 F_2}{\partial z^2}-\frac{F_2}{r^2}+\frac{1}{r}\frac{\partial F_2}{\partial r}+\frac{\partial^2 F_2}{\partial r^2}+\frac{1}{r^2}\frac{\partial F_1}{\partial\theta}-\cancel{\frac{1}{r}\frac{\partial^2 F_1}{\partial r\partial\theta}}\right]$$

$$=\left(\frac{1}{r}\frac{\partial F_2}{\partial r}+\frac{\partial^2 F_2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_2}{\partial\theta^2}+\frac{\partial^2 F_2}{\partial z^2}\right)-\frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial\theta}$$

$$=\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F_2}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F_2}{\partial\theta^2}+\frac{\partial^2 F_2}{\partial z^2}\right)-\frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial\theta}$$

$$=\nabla^2 F_2-\frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial\theta}$$

$c=$

$$=\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]$$

$$-\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)\right]$$

$$=\frac{\partial}{\partial z}\left[\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_1)+\frac{\partial}{\partial\theta}(F_2)+\frac{\partial}{\partial z}(rF_3)\right]\right]$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\left[\frac{\partial}{\partial z}(F_1)-\frac{\partial}{\partial r}(F_3)\right]\right)-\frac{\partial}{\partial\theta}\left(\frac{1}{r}\left[\frac{\partial}{\partial\theta}(F_3)-\frac{\partial}{\partial z}(rF_2)\right]\right)\right]$$

$$=\frac{\partial}{\partial z}\left[\frac{1}{r}\left[F_1+r\frac{\partial F_1}{\partial r}+\frac{\partial F_2}{\partial\theta}+r\frac{\partial F_3}{\partial z}\right]\right]$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial F_1}{\partial z}-r\frac{\partial F_3}{\partial r}\right)-\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial F_3}{\partial\theta}-\frac{\partial F_2}{\partial z}\right)\right]$$

$$=\frac{\partial}{\partial z}\left[\frac{F_1}{r}+\frac{\partial F_1}{\partial r}+\frac{1}{r}\frac{\partial F_2}{\partial\theta}+\frac{\partial F_3}{\partial z}\right]$$

$$-\frac{1}{r}\left[\left(\frac{\partial F_1}{\partial z}+r\frac{\partial^2 F_1}{\partial r\partial z}-\frac{\partial F_3}{\partial r}-r\frac{\partial^2 F_3}{\partial r^2}\right)-\left(\frac{1}{r}\frac{\partial^2 F_3}{\partial\theta^2}-\frac{\partial^2 F_2}{\partial\theta\partial z}\right)\right]$$

$$=\cancel{\frac{1}{r}\frac{\partial F_1}{\partial z}}+\cancel{\frac{\partial^2 F_1}{\partial z\partial r}}+\cancel{\frac{1}{r}\frac{\partial^2 F_2}{\partial z\partial\theta}}+\frac{\partial^2 F_3}{\partial z^2}$$

$$+\left[-\cancel{\frac{1}{r}\frac{\partial F_1}{\partial z}}-\cancel{\frac{\partial^2 F_1}{\partial r\partial z}}+\frac{1}{r}\frac{\partial F_3}{\partial r}+\frac{\partial^2 F_3}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_3}{\partial\theta^2}-\cancel{\frac{1}{r}\frac{\partial^2 F_2}{\partial\theta\partial z}}\right]$$

$$=\frac{1}{r}\frac{\partial F_3}{\partial r}+\frac{\partial^2 F_3}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_3}{\partial\theta^2}+\frac{\partial^2 F_3}{\partial z^2}$$

$$=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F_3}{\partial r}\right)+\frac{\partial^2 F_3}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_3}{\partial\theta^2}+\frac{\partial^2 F_3}{\partial z^2}$$

$$=\nabla^2 F_3$$

3. Derivatives of the unit vectors in orthogonal curvilinear coordinate systems

The last topic to be discussed concerning curvilinear coordinates is the procedure to obtain the derivatives of the unit vectors, i.e. $\frac{\partial}{\partial x_j}\hat{\mathbf{e}}_i = \hat{\mathbf{e}}_{ij}$

Such quantities are required, for example, in obtaining the rate-of-strain and rotation tensor

$$e_{ij} = \nabla\mathbf{V}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(e_{ij} + e_{ij}^T\right) = \frac{1}{2}\left(\nabla\mathbf{V} + \mathbf{V}\nabla\right)$$

$$\omega_{ij} = \frac{1}{2}\left(e_{ij} - e_{ij}^T\right) = \frac{1}{2}\left(\nabla\mathbf{V} - \mathbf{V}\nabla\right)$$

To simplify the notation we define:

$$\mathbf{R}_{x_i} = \mathbf{r}_i, \quad \mathbf{R}_{x_i} = h_i\hat{\mathbf{e}}_i = \mathbf{r}_i$$

$$\frac{\partial}{\partial x_j}\mathbf{R}_{x_i} = \mathbf{r}_{ij} \text{ and } \frac{\partial}{\partial x_j}h_i = h_{ij}$$

Note that $\mathbf{r}_{ij}$ is symmetric, i.e. $\mathbf{r}_{ij} = \mathbf{r}_{ji}$

$$\mathbf{r}_1 = h_1\hat{\mathbf{e}}_1$$
$$\mathbf{r}_2 = h_2\hat{\mathbf{e}}_2$$
$$\mathbf{r}_3 = h_3\hat{\mathbf{e}}_3$$
$$\mathbf{r}_{11} = a\hat{\mathbf{e}}_1 + b\hat{\mathbf{e}}_2 + c\hat{\mathbf{e}}_3 = h_{11}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{11}$$
$$\mathbf{r}_{12} = h_{12}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{12}$$
$$\mathbf{r}_{13} = h_{13}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{13}$$

3.1 Derivation of $\hat{\mathbf{e}}_{11} = -\frac{h_{12}}{h_2}\hat{\mathbf{e}}_2 - \frac{h_{13}}{h_3}\hat{\mathbf{e}}_3$

$$\mathbf{r}_1 \cdot \mathbf{r}_1 = h_1^2$$
$$\mathbf{r}_1 \cdot \mathbf{r}_{11} = h_1h_{11}$$
$$\mathbf{r}_1 \cdot \mathbf{r}_{12} = h_1h_{12}$$
$$\mathbf{r}_1 \cdot \mathbf{r}_{13} = h_1h_{13}$$

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$$
$$\rightarrow \frac{\partial\left(\mathbf{r}_1 \cdot \mathbf{r}_2\right)}{\partial x_1} = 0$$
$$\rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_{21} = 0$$
$$\rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_2 = -\mathbf{r}_1 \cdot \mathbf{r}_{12}$$

$$\rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_2 = -h_1 h_{12}$$

$$\mathbf{r}_1 \cdot \mathbf{r}_3 = 0$$
$$\rightarrow \frac{\partial(\mathbf{r}_1 \cdot \mathbf{r}_3)}{\partial x_1} = 0$$
$$\rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_3 + \mathbf{r}_1 \cdot \mathbf{r}_{31} = 0$$
$$\rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_3 = -\mathbf{r}_1 \cdot \mathbf{r}_{13}$$
$$\rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_3 = -h_1 h_{13}$$

$$\mathbf{r}_{11} = h_{11}\hat{\mathbf{e}}_1 - \frac{h_1 h_{12}}{h_2}\hat{\mathbf{e}}_2 - \frac{h_1 h_{13}}{h_3}\hat{\mathbf{e}}_3 = h_{11}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{11}$$
$$\rightarrow \hat{\mathbf{e}}_{11} = -\frac{h_{12}}{h_2}\hat{\mathbf{e}}_2 - \frac{h_{13}}{h_3}\hat{\mathbf{e}}_3$$

3.2 Derivation of $\hat{\mathbf{e}}_{22} = -\frac{h_{21}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_{23}}{h_3}\hat{\mathbf{e}}_3$

$$\mathbf{r}_2 \cdot \mathbf{r}_2 = h_2^2$$
$$\mathbf{r}_2 \cdot \mathbf{r}_{22} = h_2 h_{22}$$
$$\mathbf{r}_2 \cdot \mathbf{r}_{21} = h_2 h_{21}$$
$$\mathbf{r}_2 \cdot \mathbf{r}_{23} = h_2 h_{23}$$

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$$
$$\rightarrow \frac{\partial(\mathbf{r}_1 \cdot \mathbf{r}_2)}{\partial x_2} = 0$$
$$\rightarrow \mathbf{r}_{12} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_{22} = 0$$
$$\rightarrow \mathbf{r}_{22} \cdot \mathbf{r}_1 = -\mathbf{r}_2 \cdot \mathbf{r}_{21}$$
$$\rightarrow \mathbf{r}_{22} \cdot \mathbf{r}_1 = -h_2 h_{21}$$

$$\mathbf{r}_2 \cdot \mathbf{r}_3 = 0$$
$$\rightarrow \frac{\partial(\mathbf{r}_2 \cdot \mathbf{r}_3)}{\partial x_2} = 0$$
$$\rightarrow \mathbf{r}_{22} \cdot \mathbf{r}_3 + \mathbf{r}_2 \cdot \mathbf{r}_{32} = 0$$
$$\rightarrow \mathbf{r}_{22} \cdot \mathbf{r}_3 = -\mathbf{r}_2 \cdot \mathbf{r}_{23}$$
$$\rightarrow \mathbf{r}_{22} \cdot \mathbf{r}_3 = -h_2 h_{23}$$

$$\mathbf{r}_{22} = -\frac{h_2 h_{21}}{h_1}\hat{\mathbf{e}}_1 + h_{22}\hat{\mathbf{e}}_2 - \frac{h_2 h_{23}}{h_3}\hat{\mathbf{e}}_3 = h_{22}\hat{\mathbf{e}}_2 + h_2\hat{\mathbf{e}}_{22}$$

$$\rightarrow \hat{\mathbf{e}}_{22} = -\frac{h_{21}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_{23}}{h_3}\hat{\mathbf{e}}_3$$

3.3 Derivation of $\hat{\mathbf{e}}_{33} = -\frac{h_{31}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_{32}}{h_2}\hat{\mathbf{e}}_2$

$$\mathbf{r}_3 \cdot \mathbf{r}_3 = h_3^2$$
$$\mathbf{r}_3 \cdot \mathbf{r}_{31} = h_3 h_{31}$$
$$\mathbf{r}_3 \cdot \mathbf{r}_{32} = h_3 h_{32}$$
$$\mathbf{r}_3 \cdot \mathbf{r}_{33} = h_3 h_{33}$$

$$\mathbf{r}_1 \cdot \mathbf{r}_3 = 0$$
$$\rightarrow \frac{\partial(\mathbf{r}_1 \cdot \mathbf{r}_3)}{\partial x_3} = 0$$
$$\rightarrow \mathbf{r}_{13} \cdot \mathbf{r}_3 + \mathbf{r}_1 \cdot \mathbf{r}_{33} = 0$$
$$\rightarrow \mathbf{r}_{33} \cdot \mathbf{r}_1 = -\mathbf{r}_3 \cdot \mathbf{r}_{31}$$
$$\rightarrow \mathbf{r}_{33} \cdot \mathbf{r}_1 = -h_3 h_{31}$$

$$\mathbf{r}_2 \cdot \mathbf{r}_3 = 0$$
$$\rightarrow \frac{\partial(\mathbf{r}_2 \cdot \mathbf{r}_3)}{\partial x_3} = 0$$
$$\rightarrow \mathbf{r}_{23} \cdot \mathbf{r}_3 + \mathbf{r}_2 \cdot \mathbf{r}_{33} = 0$$
$$\rightarrow \mathbf{r}_{33} \cdot \mathbf{r}_2 = -\mathbf{r}_3 \cdot \mathbf{r}_{32}$$
$$\rightarrow \mathbf{r}_{33} \cdot \mathbf{r}_2 = -h_3 h_{32}$$

$$\mathbf{r}_{33} = -\frac{h_3 h_{31}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_3 h_{32}}{h_2}\hat{\mathbf{e}}_2 + h_{33}\hat{\mathbf{e}}_3 = h_{33}\hat{\mathbf{e}}_3 + h_3\hat{\mathbf{e}}_{33}$$
$$\rightarrow \hat{\mathbf{e}}_{33} = -\frac{h_{31}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_{32}}{h_2}\hat{\mathbf{e}}_2$$

3.4 Derivation of $\hat{\mathbf{e}}_{32} = \frac{h_{23}}{h_3}\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_{23} = \frac{h_{32}}{h_2}\hat{\mathbf{e}}_3$

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 0 \rightarrow \frac{\partial}{\partial x_3}(\mathbf{r}_1 \cdot \mathbf{r}_2) = \mathbf{r}_{13} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_{23} = 0 \qquad (1)$$
$$\mathbf{r}_2 \cdot \mathbf{r}_3 = 0 \rightarrow \frac{\partial}{\partial x_1}(\mathbf{r}_2 \cdot \mathbf{r}_3) = \mathbf{r}_{21} \cdot \mathbf{r}_3 + \mathbf{r}_2 \cdot \mathbf{r}_{31} = 0 \qquad (2)$$
$$\mathbf{r}_3 \cdot \mathbf{r}_1 = 0 \rightarrow \frac{\partial}{\partial x_2}(\mathbf{r}_3 \cdot \mathbf{r}_1) = \mathbf{r}_{32} \cdot \mathbf{r}_1 + \mathbf{r}_3 \cdot \mathbf{r}_{12} = 0 \qquad (3)$$

$$(3)-(2)=0$$
$$\rightarrow \mathbf{r}_{32}\cdot\mathbf{r}_1-\mathbf{r}_2\cdot\mathbf{r}_{31}=0$$
$$\rightarrow \mathbf{r}_{32}\cdot\mathbf{r}_1=\mathbf{r}_2\cdot\mathbf{r}_{31}$$
$$\rightarrow \mathbf{r}_{32}\cdot\mathbf{r}_1=-\mathbf{r}_1\cdot\mathbf{r}_{23}$$
$$\rightarrow \mathbf{r}_{32}\cdot\mathbf{r}_1=0$$

$$\mathbf{r}_{32}=a\hat{\mathbf{e}}_1+b\hat{\mathbf{e}}_2+c\hat{\mathbf{e}}_3=h_{32}\hat{\mathbf{e}}_3+h_3\hat{\mathbf{e}}_{32}$$
$$\mathbf{r}_{32}=a\hat{\mathbf{e}}_1+b\hat{\mathbf{e}}_2+c\hat{\mathbf{e}}_3=h_{23}\hat{\mathbf{e}}_2+h_{32}\hat{\mathbf{e}}_3$$
$$\hat{\mathbf{e}}_{32}=\frac{h_{23}}{h_3}\hat{\mathbf{e}}_2$$

$$\mathbf{r}_{23}=h_{23}\hat{\mathbf{e}}_2+h_2\hat{\mathbf{e}}_{23}$$
$$\mathbf{r}_{23}=\mathbf{r}_{32}=a\hat{\mathbf{e}}_1+b\hat{\mathbf{e}}_2+c\hat{\mathbf{e}}_3=h_{23}\hat{\mathbf{e}}_2+h_{32}\hat{\mathbf{e}}_3$$
$$\hat{\mathbf{e}}_{23}=\frac{h_{32}}{h_2}\hat{\mathbf{e}}_3$$

3.5 Derivation of $\hat{\mathbf{e}}_{21}=\frac{h_{12}}{h_2}\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_{12}=\frac{h_{21}}{h_1}\hat{\mathbf{e}}_2$

$$(2)-(1)=0$$
$$\rightarrow \mathbf{r}_{21}\cdot\mathbf{r}_3-\mathbf{r}_1\cdot\mathbf{r}_{23}=0$$
$$\rightarrow \mathbf{r}_{21}\cdot\mathbf{r}_3=\mathbf{r}_1\cdot\mathbf{r}_{23}$$
$$\rightarrow \mathbf{r}_{21}\cdot\mathbf{r}_3=\mathbf{r}_3\cdot\mathbf{r}_{12}$$
$$\rightarrow \mathbf{r}_{21}\cdot\mathbf{r}_3=0$$

$$\mathbf{r}_{21}=h_{21}\hat{\mathbf{e}}_2+h_2\hat{\mathbf{e}}_{21}$$
$$\mathbf{r}_{21}=h_{12}\hat{\mathbf{e}}_1+h_{21}\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_{21}=\frac{h_{12}}{h_2}\hat{\mathbf{e}}_1$$

$$\mathbf{r}_{12}=h_{12}\hat{\mathbf{e}}_1+h_1\hat{\mathbf{e}}_{12}$$
$$\mathbf{r}_{12}=\mathbf{r}_{21}=h_{12}\hat{\mathbf{e}}_1+h_{21}\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_{12}=\frac{h_{21}}{h_1}\hat{\mathbf{e}}_2$$

3.6 Derivation of $\hat{\mathbf{e}}_{13}=\frac{h_{31}}{h_1}\hat{\mathbf{e}}_3$, $\hat{\mathbf{e}}_{31}=\frac{h_{13}}{h_3}\hat{\mathbf{e}}_1$

$$(3)-(1)=0$$

$$\rightarrow \mathbf{r}_3 \cdot \mathbf{r}_{12} - \mathbf{r}_{13} \cdot \mathbf{r}_2 = 0$$
$$\rightarrow \mathbf{r}_2 \cdot \mathbf{r}_{13} = \mathbf{r}_3 \cdot \mathbf{r}_{12}$$
$$\rightarrow \mathbf{r}_2 \cdot \mathbf{r}_{13} = -\mathbf{r}_2 \cdot \mathbf{r}_{31}$$
$$\rightarrow \mathbf{r}_2 \cdot \mathbf{r}_{13} = 0$$

$$\mathbf{r}_{13} = h_{13}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{13}$$
$$\mathbf{r}_{13} = h_{13}\hat{\mathbf{e}}_1 + h_{31}\hat{\mathbf{e}}_3$$
$$\hat{\mathbf{e}}_{13} = \frac{h_{31}}{h_1}\hat{\mathbf{e}}_3$$

$$\mathbf{r}_{31} = h_{31}\hat{\mathbf{e}}_3 + h_3\hat{\mathbf{e}}_{31}$$
$$\mathbf{r}_{31} = \mathbf{r}_{13} = h_{13}\hat{\mathbf{e}}_1 + h_{31}\hat{\mathbf{e}}_3$$
$$\hat{\mathbf{e}}_{31} = \frac{h_{13}}{h_3}\hat{\mathbf{e}}_1$$

4. Incompressible N-S equations in orthogonal curvilinear coordinate systems

4.1 Continuity equation $\nabla \cdot \mathbf{V} = 0$

Since $\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(h_2 h_3 F_1\right) + \frac{\partial}{\partial x_2}\left(h_3 h_1 F_2\right) + \frac{\partial}{\partial x_3}\left(h_1 h_2 F_3\right)\right]$

and $\mathbf{V} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(h_2 h_3 v_1\right) + \frac{\partial}{\partial x_2}\left(h_3 h_1 v_2\right) + \frac{\partial}{\partial x_3}\left(h_1 h_2 v_3\right)\right] = 0$$

4.2 Momentum equation $\frac{\partial \mathbf{V}}{\partial t} + \left(\mathbf{V} \cdot \nabla\right)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{V}$, (where p piezometric pressure)

Since $\mathbf{V} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$, we can expand the momentum equation term by term

Local derivative $\frac{\partial \mathbf{V}}{\partial t} = \frac{\partial v_1}{\partial t}\hat{\mathbf{e}}_1 + \frac{\partial v_2}{\partial t}\hat{\mathbf{e}}_2 + \frac{\partial v_3}{\partial t}\hat{\mathbf{e}}_3$

Convective derivative $\left(\mathbf{V} \cdot \nabla\right)\mathbf{V}$

Since $\mathbf{V} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$ and $\mathbf{V} \cdot \nabla = \frac{v_1}{h_1}\frac{\partial}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial}{\partial x_3}$

$$\left(\mathbf{V} \cdot \nabla\right)\mathbf{V} = \left(\mathbf{V} \cdot \nabla\right)\left(v_1\hat{\mathbf{e}}_1\right) + \left(\mathbf{V} \cdot \nabla\right)\left(v_2\hat{\mathbf{e}}_2\right) + \left(\mathbf{V} \cdot \nabla\right)\left(v_3\hat{\mathbf{e}}_3\right)$$
$$\left(\mathbf{V} \cdot \nabla\right)\left(v_1\hat{\mathbf{e}}_1\right) =$$

$$
\begin{aligned}
&= \frac{v_1}{h_1}\frac{\partial\left(v_1\hat{\mathbf{e}}_1\right)}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial\left(v_1\hat{\mathbf{e}}_1\right)}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial\left(v_1\hat{\mathbf{e}}_1\right)}{\partial x_3} \\
&= \frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{v_1 v_1}{h_1}\frac{\partial \hat{\mathbf{e}}_1}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}\hat{\mathbf{e}}_1 + \frac{v_2 v_1}{h_2}\frac{\partial \hat{\mathbf{e}}_1}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\hat{\mathbf{e}}_1 + \frac{v_3 v_1}{h_3}\frac{\partial \hat{\mathbf{e}}_1}{\partial x_3} \\
&= \left(\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\right)\hat{\mathbf{e}}_1 + \left(\frac{v_1 v_1}{h_1}\hat{\mathbf{e}}_{11} + \frac{v_2 v_1}{h_2}\hat{\mathbf{e}}_{12} + \frac{v_3 v_1}{h_3}\hat{\mathbf{e}}_{13}\right) \\
&= \left(\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\right)\hat{\mathbf{e}}_1 + \frac{v_1 v_1}{h_1}\left(-\frac{h_{12}}{h_2}\hat{\mathbf{e}}_2 - \frac{h_{13}}{h_3}\hat{\mathbf{e}}_3\right) \\
&\quad + \frac{v_2 v_1}{h_2}\left(\frac{h_{21}}{h_1}\hat{\mathbf{e}}_2\right) + \frac{v_3 v_1}{h_3}\left(\frac{h_{31}}{h_1}\hat{\mathbf{e}}_3\right) \\
&= \left(\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\right)\hat{\mathbf{e}}_1 + \left(\frac{v_2 v_1 h_{21}}{h_1 h_2} - \frac{v_1 v_1 h_{12}}{h_1 h_2}\right)\hat{\mathbf{e}}_2 + \left(\frac{v_3 v_1 h_{31}}{h_3 h_1} - \frac{v_1 v_1 h_{13}}{h_3 h_1}\right)\hat{\mathbf{e}}_3
\end{aligned}
$$

$$
\begin{aligned}
&\left(\mathbf{V}\cdot\nabla\right)\left(v_2\hat{\mathbf{e}}_2\right) = \\
&\quad= \frac{v_1}{h_1}\frac{\partial\left(v_2\hat{\mathbf{e}}_2\right)}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial\left(v_2\hat{\mathbf{e}}_2\right)}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial\left(v_2\hat{\mathbf{e}}_2\right)}{\partial x_3} \\
&\quad= \frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}\hat{\mathbf{e}}_2 + \frac{v_1 v_2}{h_1}\frac{\partial \hat{\mathbf{e}}_2}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{v_2 v_2}{h_2}\frac{\partial \hat{\mathbf{e}}_2}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\hat{\mathbf{e}}_2 + \frac{v_3 v_2}{h_3}\frac{\partial \hat{\mathbf{e}}_2}{\partial x_3} \\
&\quad= \left(\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\right)\hat{\mathbf{e}}_2 + \frac{v_1 v_2}{h_1}\hat{\mathbf{e}}_{21} + \frac{v_2 v_2}{h_2}\hat{\mathbf{e}}_{22} + \frac{v_3 v_2}{h_3}\hat{\mathbf{e}}_{23} \\
&\quad= \left(\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\right)\hat{\mathbf{e}}_2 + \frac{v_1 v_2}{h_1}\left(\frac{h_{12}}{h_2}\hat{\mathbf{e}}_1\right) \\
&\quad\quad + \frac{v_2 v_2}{h_2}\left(-\frac{h_{21}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_{23}}{h_3}\hat{\mathbf{e}}_3\right) + \frac{v_3 v_2}{h_3}\left(\frac{h_{32}}{h_2}\hat{\mathbf{e}}_3\right) \\
&\quad= \left(\frac{v_1 v_2 h_{12}}{h_1 h_2} - \frac{v_2 v_2 h_{21}}{h_2 h_1}\right)\hat{\mathbf{e}}_1 + \left(\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\right)\hat{\mathbf{e}}_2 + \left(\frac{v_3 v_2 h_{32}}{h_2 h_3} - \frac{v_2 v_2 h_{23}}{h_2 h_3}\right)\hat{\mathbf{e}}_3
\end{aligned}
$$

$$
\begin{aligned}
&\left(\mathbf{V}\cdot\nabla\right)\left(v_3\hat{\mathbf{e}}_3\right) = \\
&\quad= \frac{v_1}{h_1}\frac{\partial\left(v_3\hat{\mathbf{e}}_3\right)}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial\left(v_3\hat{\mathbf{e}}_3\right)}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial\left(v_3\hat{\mathbf{e}}_3\right)}{\partial x_3} \\
&\quad= \frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}\hat{\mathbf{e}}_3 + \frac{v_1 v_3}{h_1}\frac{\partial \hat{\mathbf{e}}_3}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}\hat{\mathbf{e}}_3 + \frac{v_2 v_3}{h_2}\frac{\partial \hat{\mathbf{e}}_3}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\hat{\mathbf{e}}_3 + \frac{v_3 v_3}{h_3}\frac{\partial \hat{\mathbf{e}}_3}{\partial x_3} \\
&\quad= \left(\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\right)\hat{\mathbf{e}}_3 + \frac{v_1 v_3}{h_1}\hat{\mathbf{e}}_{31} + \frac{v_2 v_3}{h_2}\hat{\mathbf{e}}_{32} + \frac{v_3 v_3}{h_3}\hat{\mathbf{e}}_{33}
\end{aligned}
$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\right)\hat{\mathbf{e}}_3+\frac{v_1v_3}{h_1}\left(\frac{h_{13}}{h_3}\hat{\mathbf{e}}_1\right)+\frac{v_2v_3}{h_2}\left(\frac{h_{23}}{h_3}\hat{\mathbf{e}}_2\right)$$
$$+\frac{v_3v_3}{h_3}\left(-\frac{h_{31}}{h_1}\hat{\mathbf{e}}_1-\frac{h_{32}}{h_2}\hat{\mathbf{e}}_2\right)$$
$$=\left(\frac{v_1v_3h_{13}}{h_1h_3}-\frac{v_3v_3h_{31}}{h_3h_1}\right)\hat{\mathbf{e}}_1+\left(\frac{v_2v_3h_{23}}{h_2h_3}-\frac{v_3v_3h_{32}}{h_3h_2}\right)\hat{\mathbf{e}}_2+\left(\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\right)\hat{\mathbf{e}}_3$$

Pressure gradient $\nabla p=\frac{1}{h_1}\frac{\partial p}{\partial x_1}\hat{\mathbf{e}}_1+\frac{1}{h_2}\frac{\partial p}{\partial x_2}\hat{\mathbf{e}}_2+\frac{1}{h_3}\frac{\partial p}{\partial x_3}\hat{\mathbf{e}}_3$

Viscous term $\nabla^2\mathbf{V}$

$$\nabla^2\mathbf{V}=$$
$$=\frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]\hat{\mathbf{e}}_1$$
$$-\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2v_2)-\frac{\partial}{\partial x_2}(h_1v_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1v_1)-\frac{\partial}{\partial x_1}(h_3v_3)\right]\right)\right]\hat{\mathbf{e}}_1$$
$$+\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]\hat{\mathbf{e}}_2$$
$$-\frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3v_3)-\frac{\partial}{\partial x_3}(h_2v_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2v_2)-\frac{\partial}{\partial x_2}(h_1v_1)\right]\right)\right]\hat{\mathbf{e}}_2$$
$$+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]\hat{\mathbf{e}}_3$$
$$-\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1v_1)-\frac{\partial}{\partial x_1}(h_3v_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3v_3)-\frac{\partial}{\partial x_3}(h_2v_2)\right]\right)\right]\hat{\mathbf{e}}_3$$

4.2.1 Combine terms in $\hat{e}_1$ direction to get momentum equation in $\hat{e}_1$ direction

$$\frac{\partial v_1}{\partial t}+\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}+\frac{v_1v_2h_{12}}{h_1h_2}-\frac{v_2v_2h_{21}}{h_2h_1}+\frac{v_1v_3h_{13}}{h_1h_3}-\frac{v_3v_3h_{31}}{h_3h_1}$$
$$=-\frac{1}{\rho}\frac{1}{h_1}\frac{\partial p}{\partial x_1}$$
$$+\nu\frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]$$
$$-\nu\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2v_2)-\frac{\partial}{\partial x_2}(h_1v_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1v_1)-\frac{\partial}{\partial x_1}(h_3v_3)\right]\right)\right]$$

4.2.2 Combine terms in $\hat{e}_2$ direction to get momentum equation in $\hat{e}_2$ direction

$$\frac{\partial v_2}{\partial t}+\frac{v_2 v_1 h_{21}}{h_1 h_2}-\frac{v_1 v_1 h_{12}}{h_1 h_2}+\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}+\frac{v_2 v_3 h_{23}}{h_2 h_3}-\frac{v_3 v_3 h_{32}}{h_3 h_2}$$
$$=-\frac{1}{\rho}\frac{1}{h_2}\frac{\partial p}{\partial x_2}$$
$$+\nu\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 v_1)+\frac{\partial}{\partial x_2}(h_3 h_1 v_2)+\frac{\partial}{\partial x_3}(h_1 h_2 v_3)\right]\right]$$
$$-\nu\frac{1}{h_1 h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 v_3)-\frac{\partial}{\partial x_3}(h_2 v_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2 v_2)-\frac{\partial}{\partial x_2}(h_1 v_1)\right]\right)\right]$$

4.2.3 Combine terms in $\hat{e}_3$ direction to get momentum equation in $\hat{e}_3$ direction

$$\frac{\partial v_3}{\partial t}+\frac{v_3 v_1 h_{31}}{h_3 h_1}-\frac{v_1 v_1 h_{13}}{h_3 h_1}+\frac{v_3 v_2 h_{32}}{h_2 h_3}-\frac{v_2 v_2 h_{23}}{h_2 h_3}+\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}$$
$$=-\frac{1}{\rho}\frac{1}{h_3}\frac{\partial p}{\partial x_3}$$
$$+\nu\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 v_1)+\frac{\partial}{\partial x_2}(h_3 h_1 v_2)+\frac{\partial}{\partial x_3}(h_1 h_2 v_3)\right]\right]$$
$$-\nu\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1 h_3}\left[\frac{\partial}{\partial x_3}(h_1 v_1)-\frac{\partial}{\partial x_1}(h_3 v_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 v_3)-\frac{\partial}{\partial x_3}(h_2 v_2)\right]\right)\right]$$

5. Example: Incompressible N-S equations in cylindrical polar systems

5.1 Continuity equation $\nabla\cdot\mathbf{V}=0$ in cylindrical coordinates (r,θ,z)

For cylindrical coordinates (r,θ,z), $h_1=h_r=1,\ \ h_2=h_\theta=r,\ \ h_3=h_z=1$

$$\nabla\cdot\mathbf{V}=\frac{1}{r}\left[\frac{\partial}{\partial r}(rv_r)+\frac{\partial}{\partial\theta}(v_\theta)+\frac{\partial}{\partial z}(rv_z)\right]=0$$
$$=\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(v_\theta)+\frac{\partial}{\partial z}(v_z)=0$$

5.2 Momentum equation $\frac{\partial\mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{V}$ in cylindrical coordinates (r,θ,z)

For cylindrical coordinates (r,θ,z), $h_1=h_r=1,\ \ h_2=h_\theta=r,\ \ h_3=h_z=1$ and only $h_{21}=h_{\theta r}=1$, all others are zero.

5.2.1 The r-momentum equation:

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 v_r-\frac{1}{r^2}v_r-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$

5.2.2 The θ-momentum equation:

$$\frac{\partial v_\theta}{\partial t}+\frac{1}{r}v_r v_\theta+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}=-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+\nu\left(\nabla^2 F_2-\frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial \theta}\right)$$

5.2.3 The z-momentum equation:

$$\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 v_z$$