Physical Optics 2018/ *Dr. Muwafaq Fadhil Al-Mishlab*

8th lecture [Fraunhofer diffraction, diffraction by single slit, double-slit diffraction]

1. Introduction

Diffraction is any deviation from geometrical optics that results from the obstruction of a wavefront of light. For example, an opaque screen with round hole represents such an obstruction. Diffraction phenomena are conveniently divided into two general classes, (1) those in which the source of light and the screen on which the pattern is observed are effectively at infinite distances from the aperture causing the diffraction and (2) those in which either the source or the screen, or both, are at finite distances from the aperture. The phenomena coming under class (1) are called, for historical reasons, *Fraunhofer diffraction*, and those coming under class (2) *Fresnel diffraction*.

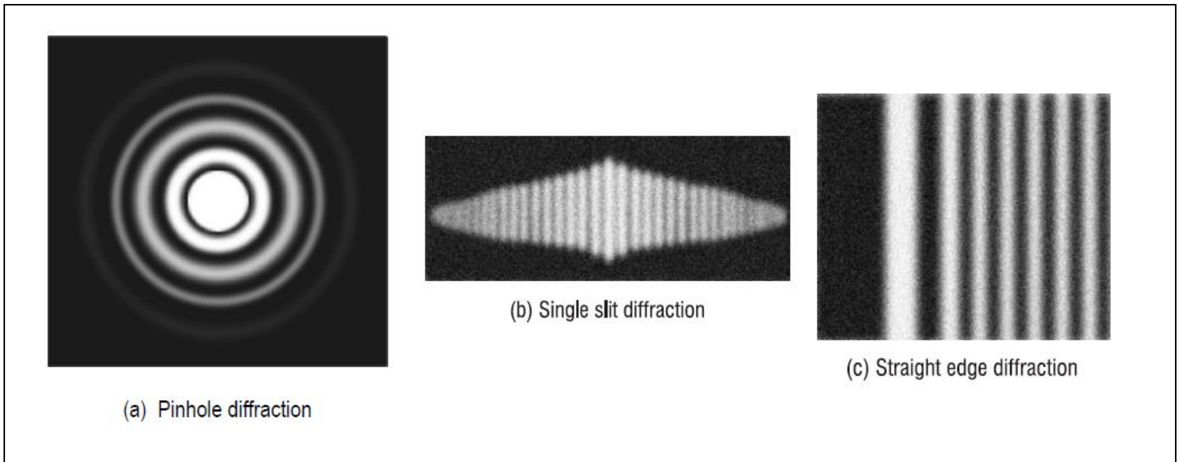


Fig. 1: Sketches of several common diffraction patterns.

2. Diffraction by A Single Slit

A rectangular aperture characterized by a length much larger than its width. The source must be far enough away, so that the wave fronts of light reaching the slit are essentially plane. This is easily accomplished in practice by placing the source in the focal plane of a positive lens. Similarly, we consider the observation screen to be effectively at infinity by using another lens on other side of the lit, as shown in figure (1). The light reaching any point such as P on the screen is due to parallel rays of light from diffraction portions of wavefront at slit (dash line). The waves do not arrive at P in phase. a ray from the center of the slit has an optical path length that is an amount Δ shorter than one leaving from a point a vertical distance S above the optical axis.

The plane portion of a wavefront at the slit opening represents a continuous array of Huygen's wavelet sources. We consider each interval of dimension ds as a source and calculate the result of all such sources by integrating over the entire slit width b. each interval ds contributes spherical wavelets at P of the form:

$$dE_p = \left(\frac{dE_0}{r}\right) e^{-i(kr-\omega t)} \text{ ------------------------ (1)}$$

Where *r* is the optical path length from the interval *ds* to the point *P*. the amplitude dE_0 is divided by *r* because the spherical waves decrease in irradiance with distance, in accordance with the inverse square law, that is, $E^2 \propto {}^1/_{r^2}\ and\ E \propto {}^1/_r$

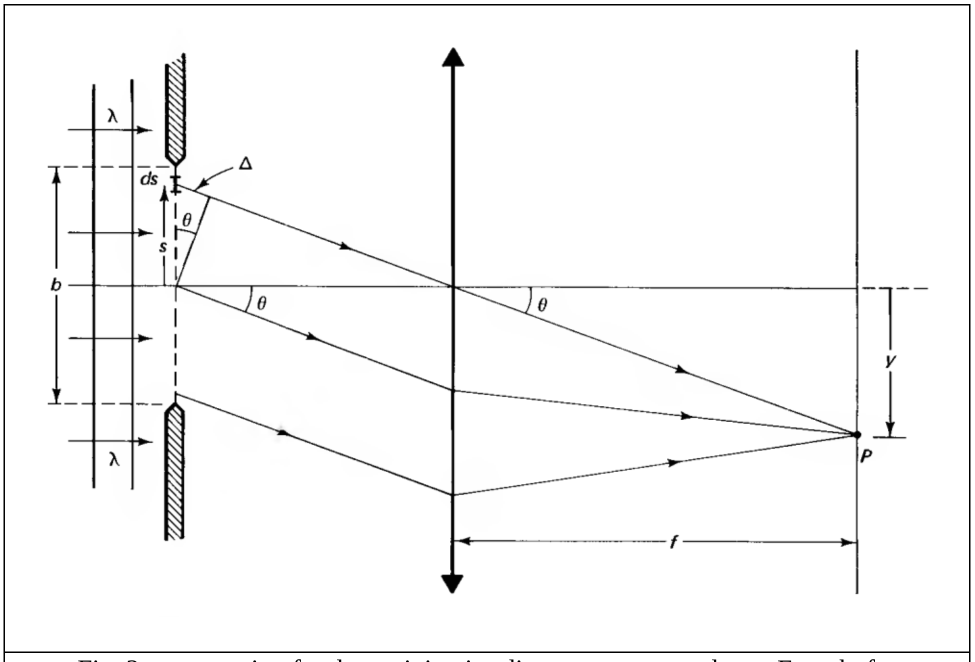


Fig. 2: construction for determining irradiance on a screen due to Fraunhofer diffraction by a single slit.

The amplitude at unit distance from the source point is then dE_0. Let us set $r = r_0$ for the wave from *ds* at *s*= 0 then for any other wave originating at the interval *ds* at height *s*, taking the difference in phase into account, the differential field at *P* is

$$dE_p = \left(\frac{dE_0}{r}\right) e^{-i[k(r_0+\Delta)-\omega t)]} \text{ ----------------------- (2)}$$

In the amplitude, $\frac{dE_0}{(r_0-\Delta)}$, the path difference Δ is unimportant, since Δ<< r_0, and therefore Δ can be neglected. The phase is very sensitive to small difference. For intervals *ds* below the axis, *s* is negative and the path difference is (r_0 – Δ), corresponding to shorter optical paths to *P*. the amplitude of the radiation from each interval clearly depends in the size of *ds*, so that when all such contributions are added by integration, we have the total effect at *P*. we write

$$dE_0 = E_L\, ds \text{ ------------------- (3)}$$

Where E_L is the amplitude per unit width of slit at unity distance away. For a point *P* at angle θ below the axis. Relative to the lens center, the figure shows that Δ= *s* sin θ. With these modifications, the differential contribution to the field at *P* from any arbitrary interval *ds*

$$dE_p = \left(\frac{E_L\, ds}{r_0}\right) e^{-i(kr_0 + k\, r \sin\theta - \omega t)}$$

Integration over the width of the slit, we have

$$E_P = \left(\frac{E_L}{r_0} \int_{-b/2}^{b/2} e^{-i\, k\, s \sin\theta}\, ds\right) e^{-i(k\, r_0 - \omega t)} \text{ ----------------------- (4)}$$

Since we are ultimately concerned with the irradiance, the square of the amplitude which we shall call E_R, we retain only the portion in parentheses and integrate:

$$E_R = \frac{E_L}{r_0} \left(\frac{e^{-ik\, s \sin\theta}}{ik \sin\theta}\right)_{-b/2}^{b/2} \text{ ---------------------------- (5)}$$

Inserting the limits of integration into Eq. 5

$$E_R = \frac{E_L}{r_0} \frac{1}{ik \sin\theta} \left[e^{(ikb \sin\theta)/2} - e^{-(ikb \sin\theta)/2}\right] \text{ ----------------------- (6)}$$

The phases of the exponential terms suggest we make a convenient substitution,

$$\beta = \frac{1}{2}\, kb \sin\theta \text{ --------------------- (7)}$$

Then

$$E_R = \frac{E_L}{r_0} \frac{b}{2i\beta} \left(e^{i\beta} - e^{-i\beta}\right) = \frac{E_L}{r_0} \frac{b}{2i\beta} (2i \sin\beta) \text{ ---------------------- (8)}$$

Where we have applied Euler's equation to the exponential terms. Simplifying,

$$E_R = \frac{E_L\, b}{r_0} \frac{\sin\beta}{\beta} \text{ -------------------- (9)}$$

The amplitude of the resultant field at *P* given by Eq. 9, where β varies with θ and thus with observation point *P* on the screen. Since phase difference is given in general by *k*Δ, Eq. 7 indicates a path difference associated with β of Δ=(b/2) sin θ, thus β represents the phase difference between waves from the center and either endpoint of the slit, where *s* =b/2. The irradiance at *P* is proportional to the square of the resultant amplitude there, or

$$I = \left(\frac{\epsilon_0\, c}{2}\right) E_R^2 = \frac{\epsilon_0\, c}{2} \left(\frac{E_L\, b}{r_0}\right)^2 \frac{sin^2\, \beta}{\beta^2} \quad \text{or}$$

$I = I_0 \left(\frac{sin^2 \beta}{\beta^2}\right) = I_0 \, sinc^2 \, (\beta)$ ----------------- (10)

Where I_0 includes all constants factors. Eqs. 9 and 10 now permit us to plot the variation of irradiance with vertical distance from the axis at the screen. The (sinc) function has the property that it approaches 1 as its argument approaches 0:

$\lim_{\beta \to 0} sinc \, (\beta) = \lim_{\beta \to 0} \left(\frac{\sin \beta}{\beta}\right) = 1$ ----------------- (11)

Otherwise, its zeros occur when sinβ=0, that is, when

$\beta = \frac{1}{2} \, (kb \sin \theta) = m\pi, with \; m = \pm 1, \pm 2, ...$

The irradiance is plotted as a function of β in figure (2). Setting $k=2\pi/\lambda$, the condition for zeros of the (sinc) function and so of the irradiance is

$m\lambda = b \sin \theta$ --------------- (12)

On the screen, therefore, the irradiance is a maximum at $\theta = 0$, or $y = 0$ and drops to zero at the values y such that $y = \frac{m\lambda f}{b} = \frac{m\lambda \, L}{b}$ -------------- (13)

The approximation in Eq. 13 comes from setting $\sin \theta = y/f$, since θ is small angle. The irradiance pattern is symmetrical about $y = 0$

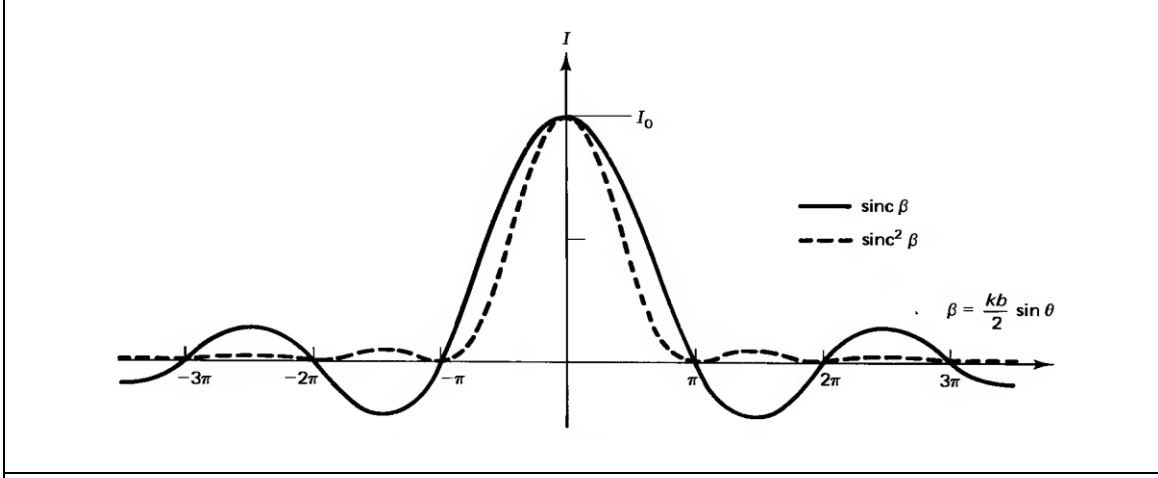


Fig. 3: Sinc function (solid line) plotted as a function of β. The irradiance function (dashed line) for single slit Fraunhofer diffraction is just the square of sinc β, normalized to I_0 at the center of the pattern.

The secondary maxima of the single-slit diffraction do not quit fall at the midpoint between zeros. The maxima coincide with maxima of the sinc function, point satisfying

$$\frac{d}{d\beta}\left(\frac{\sin\beta}{\beta}\right) = \frac{\beta\cos\beta - \sin\beta}{\beta^2} = 0$$

Example

What is the ratio of irradiance at central peak maximum to the first of the secondary maxima?

The ratio to be calculated is

$$\frac{I_{\beta=0}}{I_{\beta=1.43\pi}} = \frac{(sin^2\ \beta/\beta^2)_{\beta=0}}{(sin^2\ \beta/\beta^2)_{\beta=1.43\pi}} = \frac{1}{(sin^2\ \beta/\beta^2)_{\beta=1.43\pi}} = \left(\frac{\beta^2}{sin^2\ \beta}\right)_{1.43\pi} = \frac{20.18}{0.952} = 21.2$$

Thus, the maximum irradiance of the nearest secondary peak is only 4.7% that of the central peak.

The central maximum represents essentially the image of the slit on a distant screen. We observe that the edges of the image are not sharp but reveal a series of maxima and minima that tail off into the shadow surrounding the image. These effects are typical of the blurring of images due to diffraction. The angular width of the central maximum is defined as the angle $\Delta\theta$ between the first minima on either side. Using Eq. 12 with $m = \pm 1$ and approximating $\sin\theta$ by θ, we get

$$\Delta\theta = \frac{2\lambda}{b} \text{ --------- } (14)$$

From Eq. 14 it follows that the central maximum will spread as the slit width is narrowed. Since the length of the slit is very large compared to its width, the diffraction pattern due to points of the wave front along the length of the slit has a very small angular width and is not prominent on the screen.

3. Beam Spreading

According to Eq.14, the angular spread of the central maximum in the far field is independent of distance between aperture and screen. The linear dimensions of the diffraction pattern thus increase with distance L, as shown in figure (3), such that the width W of the central maximum is given by

$$W = L\,\Delta\theta = \frac{2\,L\,\lambda}{b} \text{ ---------------- } (15)$$

We may describe the content of Eq. 15 as a linear spread of a beam of light, originally constricted to a width b.

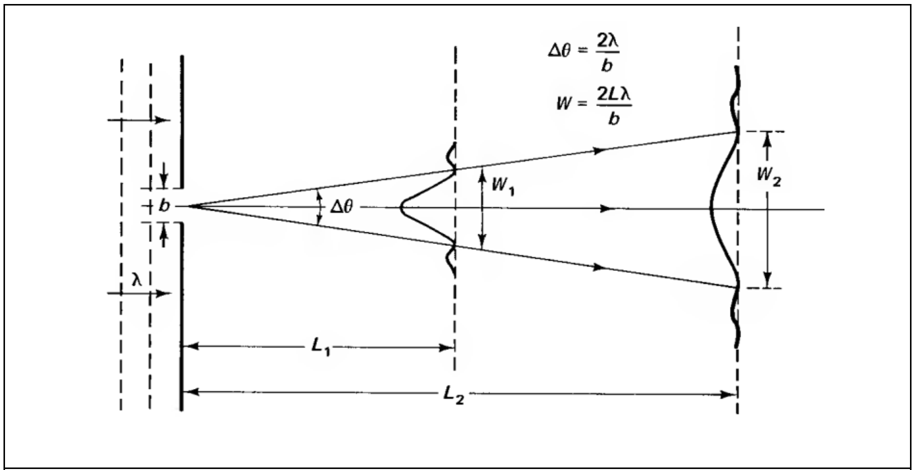


Fig. 4: Spread of the central maximum in the far-field diffraction pattern of a single-slit.

Example

Imagine a parallel beam of 546 nm light of width b = 0.5 mm propagating across the laboratory, a distance of 10 m. determines the final width of the beam due to diffraction spreading.

Solution

Using Eq. 15

$$W = L\,\Delta\theta = \frac{2\,L\,\lambda}{b} = \frac{2\,(10)(546\times10^{-9})}{0.5\times10^{-3}} = 21.8\,mm$$

4. Double-Slit Diffraction

The diffraction pattern of a plane wavefront that is obstructed everywhere except at two narrow slits is calculated in the same manner as for single slit. The mathematical argument departs from that for the single slit with Eq. 4, where limits of integration are now changed to those indicated in figure (5). Extracting the amplitude alone, we get

$$E_R = \frac{E_L}{r_0}\int_{-(\frac{1}{2})(a+b)}^{-(\frac{1}{2})(a-b)} e^{isk\sin\theta}\,ds + \frac{E_L}{r_0}\int_{(\frac{1}{2})(a-b)}^{(\frac{1}{2})(a+b)} e^{isk\sin\theta}\,ds \text{ --------------- (16)}$$

Integration and substitution of the limits leads to

$$E_R = \frac{E_L}{r_0}\frac{1}{ik\sin\theta}\left[e^{(1/2)\,ik\,(-a+b)\sin\theta} - e^{(1/2)\,ik\,(-a-b)\sin\theta} + e^{(1/2)\,ik\,(a+b)\sin\theta} - e^{(1/2)\,ik\,(a-b)\sin\theta}\right]$$

Reintroducing the substitution of Eq. 7, involving the slit width *b*,

$$\beta = \frac{1}{2}\,kb\sin\theta \text{ ---------------------- (17)}$$

and a similar one involving the slit separation *a*,

$a = \frac{1}{2}\, ka \sin\theta$ --------------------- (18)

Our equation is written more compactly as

$$E_R = \frac{E_L}{r_0}\frac{b}{2i\beta}\left[e^{ia}\left(e^{i\beta} - e^{-i\beta}\right) + e^{-ia}\left(e^{i\beta} - e^{-i\beta}\right)\right]$$

Employing Euler's equation,

$$E_R = \frac{E_L}{r_0}\frac{b}{2i\beta}\,(2i\sin\beta)\,(2\cos\alpha)$$

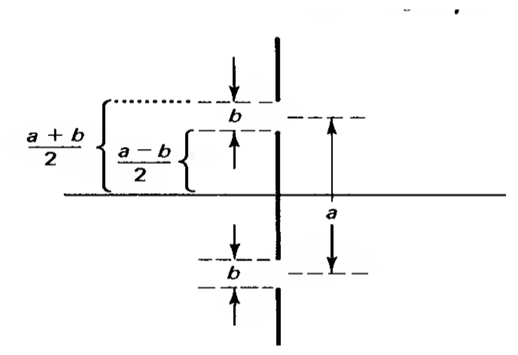


Fig. 5: Specification of slit width and separation for double-slit diffraction.

Finally,

$E_R = \frac{2E_L\,b}{r_0}\frac{\sin\beta}{\beta}\cos\alpha$ --------------------- (19)

The irradiance is now

$I = \left(\frac{\epsilon_0 c}{2}\right) E_R^2 = \left(\frac{\epsilon_0 c}{2}\right)\left(\frac{2E_L\,b}{r_0}\right)^2\left(\frac{\sin\beta}{\beta}\right)^2 cos^2\,\alpha$ ----------------- (20)

Or

$I = 4I_0\left(\frac{\sin\beta}{\beta}\right)^2 cos^2\,\alpha$ ------------------------- (21)

Where

$$I_0 = \left(\frac{\epsilon_0 c}{2}\right)\left(\frac{E_L\,b}{r_0}\right)^2$$

As defined in Eq. 10 for the single slit. Since the maximum value of Eq. 21 is $4I_0$, we see that the double slit provides four times the maximum irradiance in the pattern center as compared with the single slit. This is exactly what should be expected where the beams are in phase and amplitudes add. The factor $\left(\frac{\sin\beta}{\beta}\right)^2$ is that of Eq. 10 for single slit diffraction. The $cos^2\,\alpha$ factore, when α is written out as in Eq. 18, is

$$cos^2\,\alpha = cos^2\left[\frac{k\,a\sin\theta}{2}\right] = cos^2\left[\frac{\pi\,a\sin\theta}{\lambda}\right]$$

The diffraction envelope has a minimum when β = mπ, with m = ±1, ±2, …, as shown. In terms of the spatial angle θ, this condition is

Diffraction minimum: m λ = b sin θ --------------- (22)

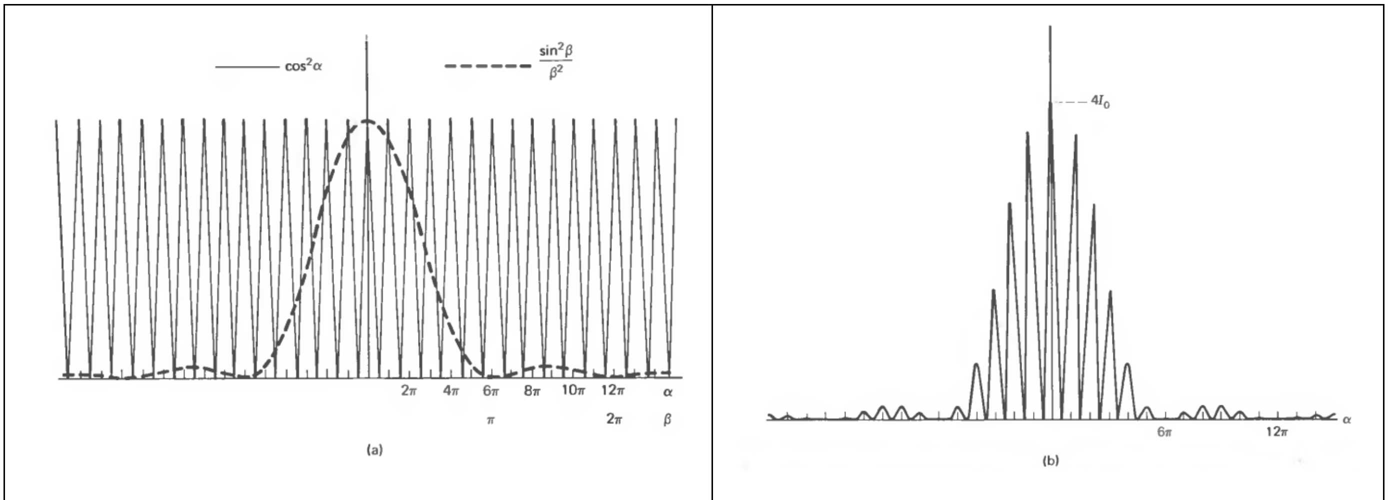
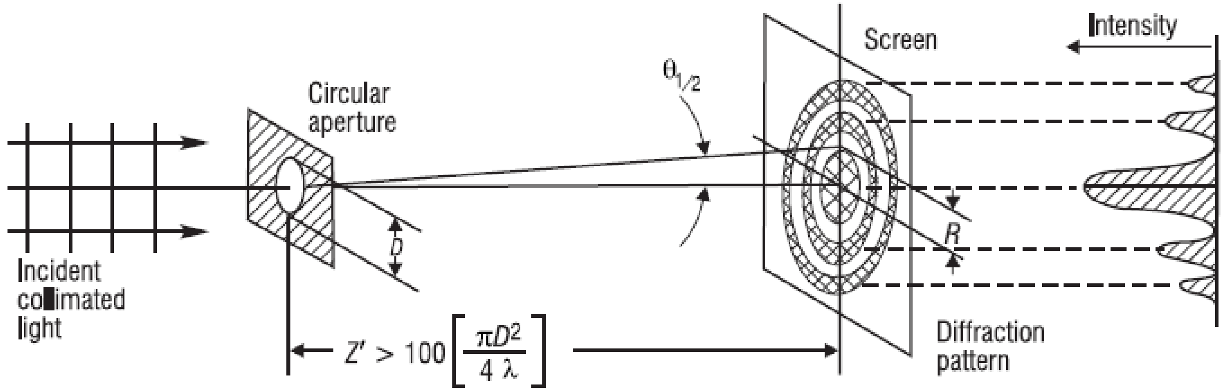


Fig. 6: a) interference (solid line) and Fraunhofer diffraction (dashed line) b) irradiance for double slit.

5. Several typical Fraunhofer diffraction patterns

In successive order, we show the far field diffraction pattern for a circular aperture (Figure 4-20), and a rectangular aperture (Figure 4-21). Equations that describe the locations of the bright and dark fringes in the patterns accompany each figure.

Circular Aperture



Half-angle beam spread to first minimum, $\theta_{1/2}$, is:

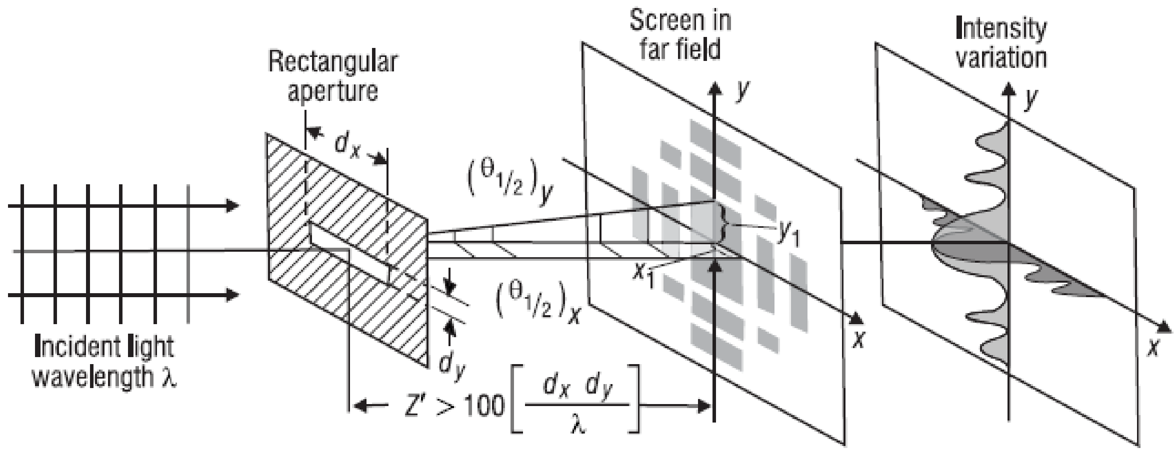
$$\theta_{1/2} = \frac{1.22\lambda}{D} \qquad (4\text{-}25)$$

Radius of central bright disk (airy disk), R, is:

$$R = \frac{1.22\,\lambda\,Z'}{D} \qquad (4\text{-}26)$$

where λ = wavelength of light,
D = diameter of pinhole, and
Z' = aperture-to-screen distance

Fig. 7: Fraunhofer diffraction pattern for a circular aperture



Half-angle beam divergences to first minimum in x and y directions:

$$\left(\theta_{1/2}\right)_x = \frac{\lambda}{d_x} \text{ and } \left(\theta_{1/2}\right)_y = \frac{\lambda}{d_y} \qquad (4\text{-}27)$$

Half-widths of central bright fringe in x and y directions:

$$x_1 = \frac{Z'\lambda}{d_x} \text{ and } y_1 = \frac{Z'\lambda}{d_y} \qquad (4\text{-}28)$$

Fig. 8: Fraunhofer diffraction pattern for a rectangular aperture.