Physical Optics 2018

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Second lecture [The Superposition of Waves]

When two sets of waves are made to cross each other, e.g., the waves created by dropping two stones simultaneously in a quiet pool, interesting and complicated effects are observed. In the region of crossing there are places where the disturbance is practically zero and others where it is greater than that given by either wave alone. A very simple law can be used to explain these effects, this is known as the *principle of superposition* and was first clearly stated by Young in 1802.

1. Superposition of Waves of The Same Frequency

Considering first the effect of superimposing two sine waves of the same frequency. The displacements due to the two waves are here taken to be along the same line, which we shall call the ***y*** direction. If the amplitudes of the two waves are $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ these will be the amplitudes of the two periodic motions impressed on the particle, we can write the separate displacements as follows:

$y_1 = a_1 \sin(\omega t - \alpha_1)\ , y_2 = a_2 \sin(\omega t - \alpha_2)$	1

Note that ω is the same for both waves, since we have assumed them to be of the same frequency. According to the principle of superposition, the resultant displacement *y* is merely the sum of y_1 and y_2, and we have:

$y = a_1 \sin(\omega t - \alpha_1) + a_2 \sin(\omega t - \alpha_2)$	2

When the expression for the sine of the difference of two angles is used,

sin (A+B) = sin A cos B + cos A sin B

$y = a_1 \sin\omega t \cos\alpha_1 - a_1 \cos\omega t \sin\alpha_1 + a_2 \sin\omega t \cos\alpha_2 - a_2 \cos\omega t \sin\alpha_2 =$ $(a_1 \cos\alpha_1 + a_2 \cos\alpha_2)\sin\omega t - (a_1 \sin\alpha_1 + a_2 \sin\alpha_2)\cos\omega t$	3

Now since the *a*'s and α's are constants,

$(a_1 \cos\alpha_1 + a_2 \cos\alpha_2) = A\cos\theta\ , (a_1 \sin\alpha_1 + a_2 \sin\alpha_2) = A\sin\theta$	4

Squaring and adding Eq. 4

$A^2(cos^2\theta + sin^2\theta) = a_1^2\,(cos^2\,\alpha_1 + sin^2\,\alpha_1) + a_2^2\,(cos^2\,\alpha_2 + sin^2\,\alpha_2) + 2a_1a_2(cos\alpha_1 \cos\alpha_2 + \sin\alpha_1 \sin\alpha_2)$	5

Or $A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\alpha_1 - \alpha_2)$	6

Dividing the lower equation (4) by the upper one, we obtain

$\tan\theta = \dfrac{a_1\sin\alpha_1 + a_2\sin\alpha_2}{a_1\cos\alpha_1 + a_2\cos\alpha_2}$	7

Equation (6) and (7) show that values of ***(A)*** and ***(θ)*** which satisfy Eqs. (4), and we can rewrite Eq. (3), substituting the right-hand members of Eq. (4). This gives

$y = A\cos\theta\sin\omega t - A\sin\theta\cos\omega t$	8

which has the form of the sine of the difference of two angles and can be expressed as

$y = A\sin(\omega t - \theta)$	9

This equation is the same as either of our original equations for the separate simple harmonic motions but contains a new amplitude ***(A)*** and a new phase constant ***(θ)***. Hence, we have the important result that the sum of two simple harmonic motions of the same frequency and along the same line is also a simple harmonic motion of the same frequency. The amplitude and phase constant of the resultant motion can easily be calculated from those of the component motions by Eqs. (6) and (7), respectively. The resultant amplitude ***A*** depends, according to Eq. (6), upon the amplitudes a_1 and a_2 of the component motions and upon their difference of phase $\delta = \alpha_1 - \alpha_2$. When we bring together two beams of light, as is done in the Michelson interferometer, the intensity of the light at any point will be proportional to the square of the resultant amplitude. By Eq. (6) we have, in the case where $a_1 = a_2$,

$I \approx A^2 = 2\,a^2\,(1 + \cos\delta) = 4\,a^2\,cos^2\,\dfrac{\delta}{2}$	10

If the phase difference is such that **$\delta = 0, 2\pi, 4\pi, \ldots$,** this gives **$4\,a^2$,** or 4 times the intensity of either beam. If **$\delta = \pi, 3\pi, 5\pi, \ldots$,** the intensity is **zero.**

2. Vector Addition of Amplitudes

very simple geometrical construction can be used to find the resultant amplitude and phase constant of the combined motion in the above case of two simple harmonic motions along the same line. If we represent the amplitudes $\mathbf{a_1}$ and $\boldsymbol{a_2}$ by vectors making angles $\boldsymbol{\alpha_1}$, and $\boldsymbol{\alpha_2}$ with x-axis as in figure (1a) the resultant amplitude A is the vector sum of $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ and makes an angle $\boldsymbol{\theta}$ with that axis.

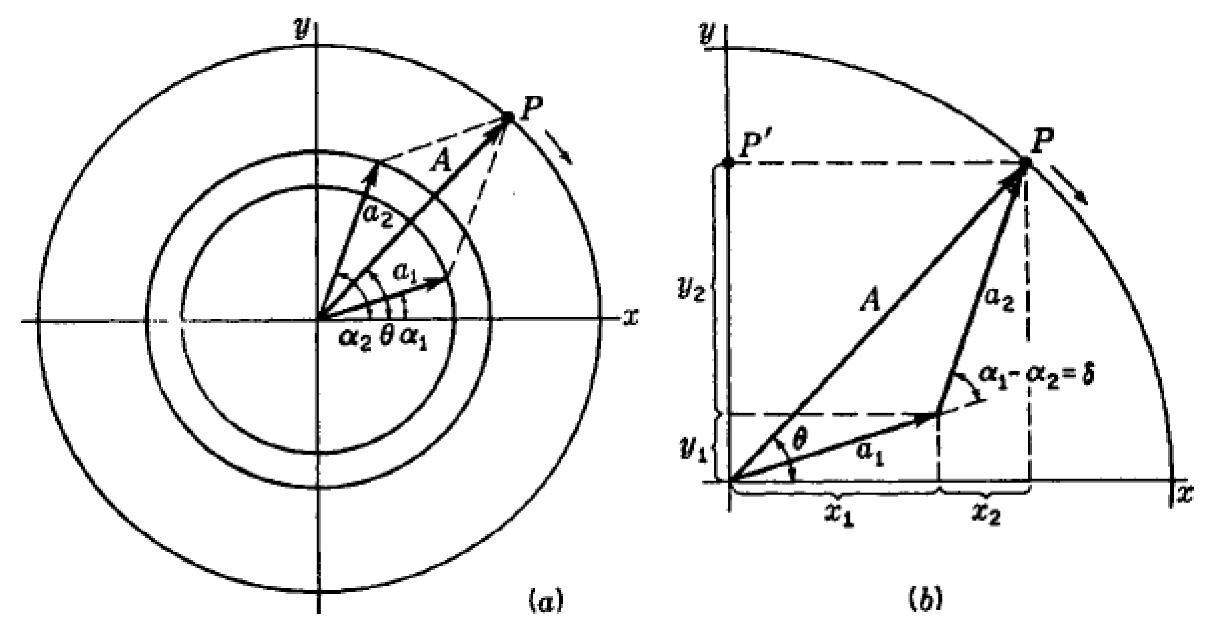

Fig. 1 Graphical composition of two waves of the same frequency, but different amplitude and phase.

To prove this, we first note from Fig. 1(b) that in the triangle formed by $\boldsymbol{a_1}$, $\boldsymbol{a_2}$ and A the law of cosines gives

$A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\pi - (\alpha_1 - \alpha_2)$	11

And the phase angle is clearly given by **Eq. (7).** The graphical method is particularly useful where we have more than two motions to compound. Figure (2) shows the result of adding five motions of equal amplitudes $\boldsymbol{a}$ and having equal phase differences $\boldsymbol{\delta}$.

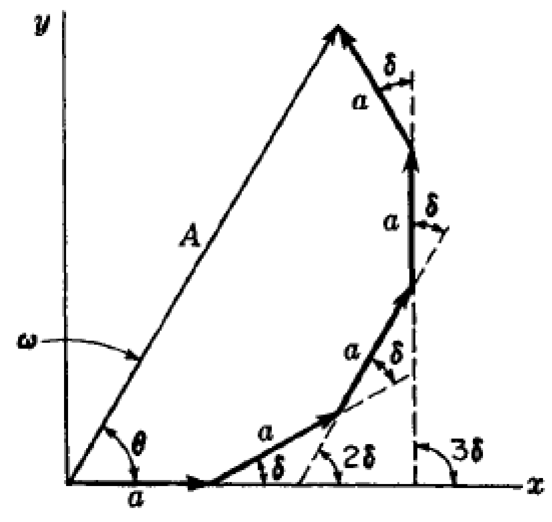

Fig. 2 Vector addition of five amplitudes having the same magnitude and phase difference ***δ***.

Example: Determine the result of superposition of the following harmonic waves:

$E_1 = 7\sin(\omega t + \pi/3)$, $E_2 = 12\cos(\omega t + \pi/4)$, $E_3 = 20\sin(\omega t + \pi/5)$

Solution:

To make all phase angles consistent, first change the cosine wave to a sine wave :

$E_2 = 12\sin(\omega t + \pi/4 + \pi/2) = 12\sin(\omega t + 3\pi/4)$ then by using following Eq.

$$E_0^2 = \left(\sum_{i=1}^{N} E_{0i}\sin\alpha_i\right)^2 + \left(\sum_{i=1}^{N} E_{0i}\cos\alpha_i\right)^2$$

$$E_0^2 = \left[7\sin\left(\frac{\pi}{3}\right) + 12\sin\left(\frac{3\pi}{4}\right) + 20\sin\left(\frac{\pi}{5}\right)\right]^2 + \left[7\cos\left(\frac{\pi}{3}\right) + 12\cos\left(\frac{3\pi}{4}\right) + 20\cos\left(\frac{\pi}{5}\right)\right]^2$$

Or $E_0^2 = (26.303)^2 + (11.195)^2$ *and* $E_0 = 28.6$

The same result can be found from the next Eq.

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2 \sum_{j>i}^{N} \sum_{i=1}^{N} E_{0i}E_{0j} \cos(\alpha_j - \alpha_i)$$

$$E_0^2 = 7^2 + 12^2 + 20^2 + 2\left[7 \times 12 \cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + 7 \times 20 \cos\left(\frac{\pi}{5} - \frac{\pi}{3}\right) + 12 \times 20 \cos\left(\frac{\pi}{5} - \frac{3\pi}{4}\right)\right]$$

The phase angle of the resulting harmonic wave is found by

$$\tan\alpha = \frac{\sum_{i=1}^{N} E_{0i} \sin\alpha_i}{\sum_{i=1}^{N} E_{0i} \cos\alpha_i}$$

$$\tan\alpha = \frac{26.303}{11.195} \quad \text{and} \quad \alpha = 1.17 \text{ (radians)}$$

Thus the resulting harmonic wave can be expressed as

$$E_0 = 28.6 \sin(\omega t + 1.17) \quad \text{or} \quad 28.6 \sin(\omega t + 0.372\pi)$$

3. Superposition of Two Wave Trains of The Same Frequency

we can conclude directly that the result of superimposing two trains of sine waves of the same frequency and traveling along the same line is to produce another sine wave of that frequency but having a new amplitude which is determined for given values of a_1 and a_2 by the phase difference δ between the motions imparted to any particle by the two waves. As an example, let us find the resultant wave produced by two waves of equal frequency and amplitude traveling in the same direction $+x$, but with one a distance Δ ahead of the other. The equations of the two waves will be

$y_1 = a\sin(\omega t - kx)\ \text{and}\ y_2 = a\sin[\omega t - k\,(x + \Delta)]$	12

By the principle of superposition, the resultant displacement is the sum of the separate ones, so that

$y = y_1 + y_2 = a\,\{\sin(\omega t - kx) + \sin[\omega t - k(x + \Delta)]\}$	13

Applying the trigonometric formula

$\sin A + \sin B \ = 2\sin\frac{1}{2}\ (A+B)\cos\frac{1}{2}\ (A-B)$	14

We find

$y = 2\ a\cos\frac{k\Delta}{2}\sin\left[\omega t - k\ \left(x + \frac{\Delta}{2}\right)\right]$	15

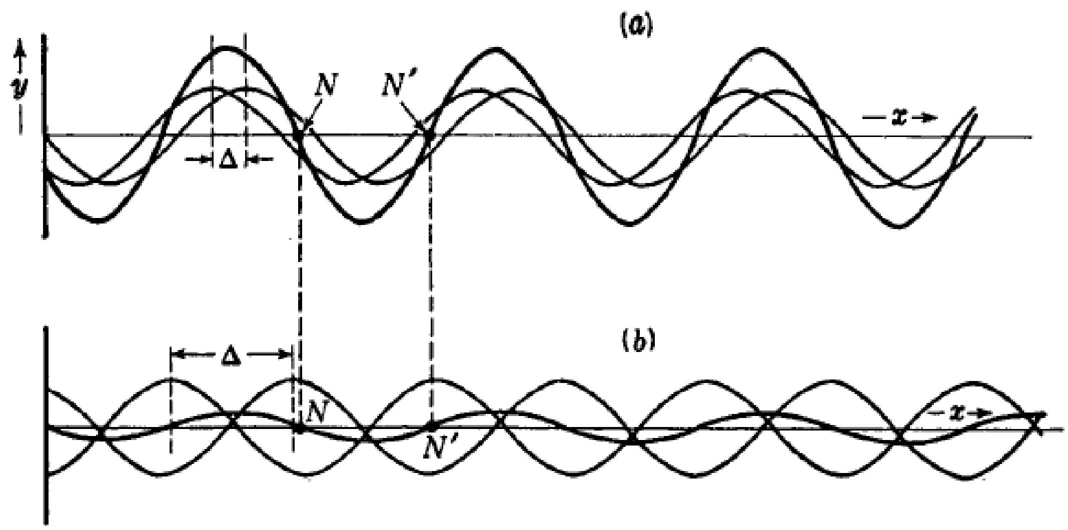

Fig. 3 Superposition of two wave trains *(a)* almost in phase and *(b)* almost 180' out of phase.

This corresponds to a new wave of the same frequency but with the amplitude ***2a cos (k Δ/2) = 2 a cos (πΔ/λ).*** When Δ is a small fraction of a wavelength, this amplitude will be nearly ***2a***, while if Δ is in the neighborhood of ½ λ, it will be practically zero. These cases are illustrated in Fig. 3 where the waves represented by Eqs. 12 (light curves) and (15) (heavy curve) are plotted at the time *t* = 0.

4. Standing Waves

Another important case of superposition arises when a given wave exists in both forward and reverse directions along the same medium. This condition occurs most frequently when the forward wave experiences a reflection at same point along its path as in figure 4.

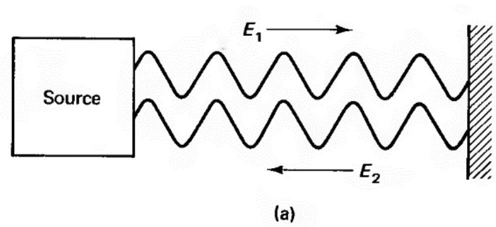

Fig. 4 A typical standing wave

Two such waves can be represented by the equations

$y_1 = a\sin(\omega t - kx)$ *and* $y_2 = a\sin(\omega t + kx)$	16

The resultant wave in the medium, by the principle of superposition, is

$y = y_1 + y_2 = a\,[\sin(\omega t + kx) + \sin(\omega t - kx)]$	17

$\alpha = \omega t + kx \text{ and } \beta = \omega t - kx$

$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta)$

$y = 2\,a\cos(-kx)\sin(\omega t)$ *or* $y = 2\,a\sin kx\cos\omega t$	18

Which represent the standing wave. For any value of x we have simple harmonic motion, whose amplitude varies with x between the limits $2a$ when $kx = 0, \pi, 2\pi, 3\pi, \ldots$ and zero when $kx = \pi/2, 3\pi/2, 5\pi/2, \ldots.$ such points are call the ***nodes*** of standing wave and are separated by half wavelength

$\boldsymbol{x = m\,(\lambda/2)}$