Physical Optics 2018

Dr. Muwafaq Fadhil Al-Mishlab

First lecture [Wave Equations]

1. One-dimensional wave equation

The most general form of a traveling wave, and the differential equation it satisfies, can be determined in the following way. Consider first a one-dimensional wave pulse of arbitrary shape, described by $y' = f(x')$, fixed to a coordinate system $O'\ (x', y')$, as in Figure 8-1a. Consider next that the O' system, together with the pulse, moves to the right along the x-axis at uniform speed v relative to a fixed coordinate system, $O(x, y)$, as in Figure 8-1b. As it moves, the pulse is assumed to maintain its shape. Any point on the pulse, such as P, can be described by either of two coordinates, x or x', where $x' = x - vt$. The y-coordinate is identical in either system. From the point of view of the stationary coordinate system, then, the moving pulse has the mathematical form

$$y = y' = f(x') = f(x - vt)$$

If the pulse moves to the left, the sign of v must be reversed, so that in general we may write

$y = f(x \pm vt)$	1

as the general form of a traveling wave. Notice that we have assumed $x = x'$ at $t = 0$. The original shape of the pulse, $y' = f(x')$, does not vary but is found simply translated along the x-direction by the amount vt at time t. The function f is any function whatsoever, so that for example,

$$y = A \sin (x - vt)$$

$$y = A(x + vt)^2$$

$$y = e^{(x-vt)}$$

all represent traveling waves. Only the first, however, represents the important case of a periodic wave.

We wish to find next the partial differential equation that is satisfied by all such periodic waves, regardless of the particular function f. Since y is a function of two variables, x and t, we use the chain rule of partial differentiation and write

$$y = f(x')$$

where

$$x' = x \pm vt$$

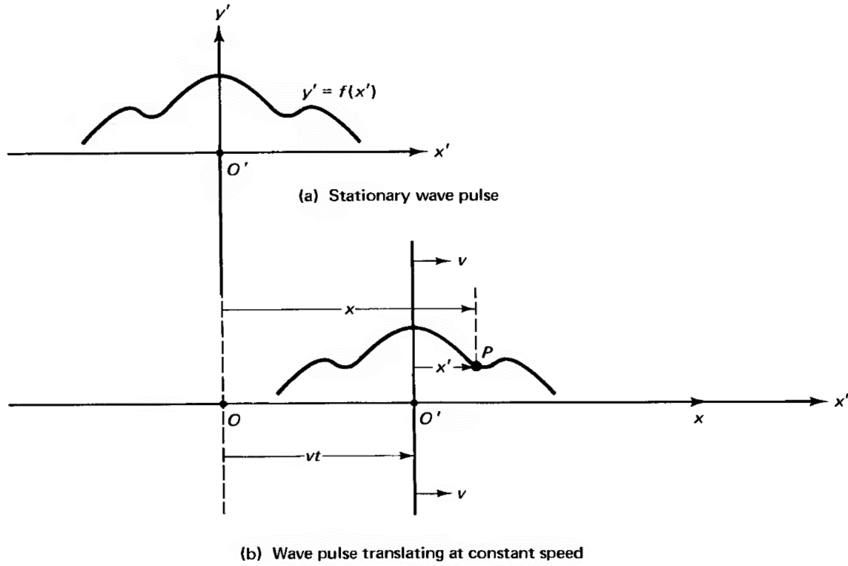

Fig. 1 Translating wave pulses.

so that

$$\partial x'/\partial x = 1 \text{ and } \partial x'/\partial t = \pm v$$

Employing the chain rule, the space derivative is

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} = \frac{\partial f}{\partial x'}$$

Repeating the procedure to find the second derivative,

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial(\partial y/\partial x)}{\partial x'}\frac{\partial x'}{\partial x} = \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x'}\right) = \frac{\partial^2 f}{\partial x'^2}$$

Similarly, the time derivatives are found:

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t} = \pm v\frac{\partial f}{\partial x'}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = \frac{\partial(\partial y/\partial t)}{\partial x'}\frac{\partial x'}{\partial t} = \frac{\partial}{\partial x'}\left(\pm v\frac{\partial f}{\partial x'}\right)(\pm v) = v^2\frac{\partial^2 f}{\partial x'^2}$$

Combining the results for the two second derivatives, we arrive at the one-dimensional differential wave equation,

$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$	2

any wave of the form of Eq. 1 must satisfy the wave Eq. 2

2. Simple Harmonic Motion

Simple harmonic motion is defined as the projection on any diameter of a graph point moving in a circle with uniform speed. The motion is illustrated in **Fig.** 2. The graph point p moves around the circle of radius a with a uniform speed v. If at every instant of time a normal is drawn to the diameter AB, the intercept P, called the mass point, moves with SHM.

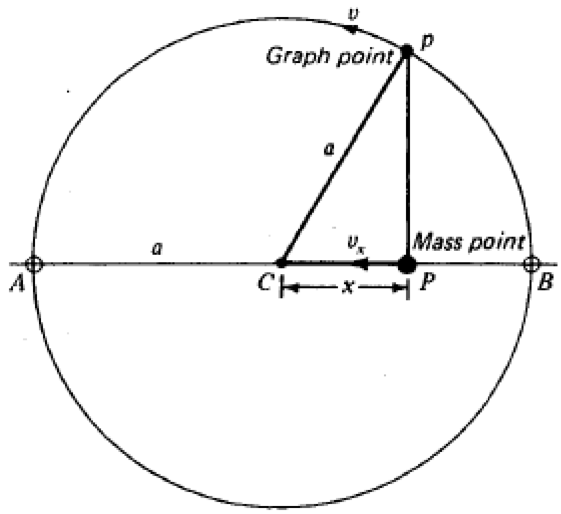

Fig. 2 Simple harmonic motion along a straight line AB.

Moving back and forth along the line AB, the mass point is continually changing speed vx' Starting from rest at the end points A or B, the speed increases until it reaches C. From there it slows down again coming to rest at the other end of its path. The return of the mass point is a repetition of this motion in reverse. The displacement of an object undergoing SHM is defined as the distance from its equilibrium position C to the point P. It will be seen in **Fig. 2** that the displacement x varies in magnitude from zero up to its maximum value a, which is the radius of the circle of reference. The maximum displacement a is called the amplitude, and the time required to make one complete vibration is called the period. If a vibration starts at B, it is completed when the mass point P moves across to A and back again to B. If it starts at C and moves to B and back to C, only half a vibration has been completed. The amplitude a is measured in meters, or a fraction thereof, while the period is measured in seconds.

The frequency of vibration is defined as the number of complete vibrations per second. If a particular vibrating body completes one vibration in t s, the period T = t s and it will make three complete vibrations in 1 s. If a body makes 10 vibrations in 1 s, its period will be T = fo s. In other words, the frequency of vibration υ and the period T are reciprocals of each other:

Frequency= 1/period , Period= 1/frequency

$\nu = \frac{1}{T}\ ,\quad T = \frac{1}{\nu}$	3

If the vibration of a body is described in terms of the graph point p, moving in a circle, the frequency is given by the number of revolutions per second, or cycles per second:

1 cycle/second = 1 vibration/ second [called Hertz , 1vib/s = 1 Hz]

3. Theory of SHM

At this point we present the theory of SHM and derive an equation for the period of vibrating bodies. In Fig.3 we see that the displacement x is given by:

$x = a\ cos\theta$	4

As the graph point *p* moves with constant speed ν, the radius vector *a* rotates with constant angular speed ω, so that the angle θ changes at a constant rate

$x = a\ cos\omega t$	5

The graph point *p*, moving with a speed ν, travels once around the circle of reference, a distance equal to 2πa, in the time of one period *T*. We now use the relation in mechanics that time equals distance divided by speed, and obtain

$\mathrm{T} = \frac{2\pi a}{v}$	6

To obtain the angular speed ω of the graph point in terms of the period, we have

$\mathrm{T} = \frac{2\pi}{\omega}\ , or\ \omega = \frac{2\pi}{\mathrm{T}}$	7

An object moving in a circle with uniform speed v has a *centripetal acceleration* toward the center, given by

$a_c = \dfrac{v^2}{a}$	8

Since this acceleration a_c continually changes the direction of the motion, its component *ax* along the diameter, or x axis, changes in magnitude and is given by $a_x = a_c \cos\theta$. Substituting in Eq. (8), we find

$a_x = \dfrac{v^2}{a}\cos\theta$	9

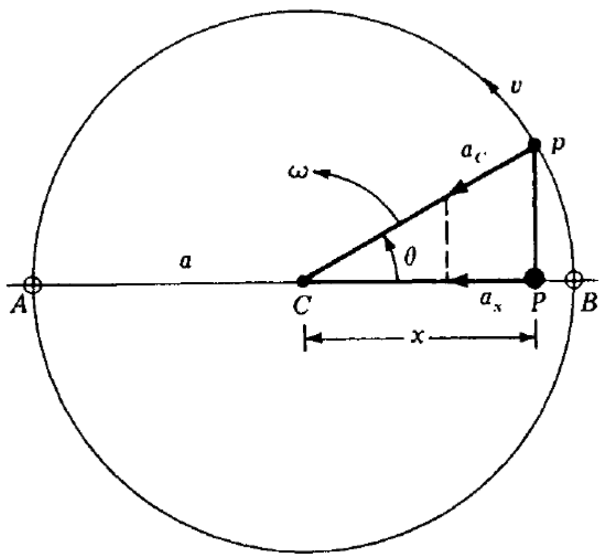

Fig. 3 The acceleration a_x of any mass moving with simple harmonic motion is toward a position of equilibrium C.

From the right triangle *CPp*, $\cos\theta = {}^{x}/_{a}$ direct substitution gives

$$a_x = \frac{v^2}{a}\frac{x}{a} \quad or \quad a_x = \frac{v^2}{a^2}\,x$$

We now multiply both sides of the equation by $a^2/a_x v^2$, take the square root of both sides of the equation, and obtain

$$\frac{a^2}{v^2} = \frac{x}{a_x} \qquad and \qquad \frac{a}{v} = \sqrt{\frac{x}{a_x}}$$

For *a/v* in Eq. (6) we now substitute $\sqrt{\frac{x}{a_x}}$ and obtain for the period of any SHM the relation

$T = 2\pi\sqrt{\frac{x}{a_x}}$	10

If the displacement is to the right of C, its value is +x, and if the acceleration is to the left, its value is –a_x. Conversely, when the displacement is to left of C, we have - x, and the acceleration is to the right, or +a_x. This is the reason for writing

$T = 2\pi\sqrt{-\frac{x}{a_x}}$	11

4. Transverse waves

All light waves are classified as *transverse waves.* Transverse waves are those in which each small part of the wave vibrates along a line perpendicular to the direction of propagation and all parts are vibrating in the same plane. When a source vibrates with SHM and sends out transverse waves through a homogeneous medium, they have the general appearance of the waves shown in **Fig. 4**.

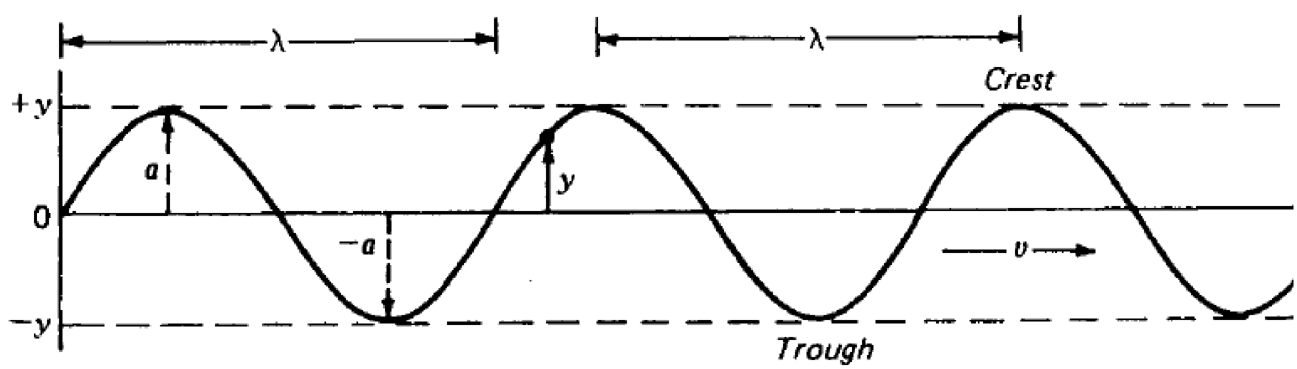

Fig. 4 Diagram of a transverse wave, vibrating in the plane of the page, showing the wavelength λ., the amplitude *a*, the displacement *y*, and the speed *v*.

The distance between two similar points of any two consecutive wave forms is called the *wavelength λ*. One wavelength, for example, is equal to the distance between two *wave crests* or two *wave troughs.* The displacement *y* of any given point along a wave, at any given instant in time, is given by the vertical distance of that point from its equilibrium position. The value is continually changing from + to - to +, etc. The amplitude of any wave is given by the letter *a* in Fig. 4, and is defined as the *maximum value of the displacement y*.

The frequency of a train of waves is given by the number of waves passing by, or arriving at, any given point per second, and is specified in *hertz,* or in vibrations per second. From the definition of frequency υ and the wavelength λ, the speed of the waves v is given by the wave equation:

$v = \upsilon\lambda$	12

The length of one wave times the number of waves per second equals the distance the waves will travel in 1 s.

5. Sine Waves

The simplest kind of wave train is that for which the motions of all points along the wave have displacements y given by the sine or cosine of some uniformly increasing function. This in effect describes what we have called SHM.

Consider transverse waves in which the motions of all parts are perpendicular to the direction of propagation. The displacement *y* of any point on the wave is then given by

$y = a\, sin \frac{2\pi x}{\lambda}$	13

A graph of this equation is shown in **Fig. 5**, and the significance of the constants *a* and *A* is clear. To make the wave move to the right with a velocity *v*, we introduce the time t as follows:

$y = a\, sin \frac{2\pi}{\lambda}\,(x - vt)$	14

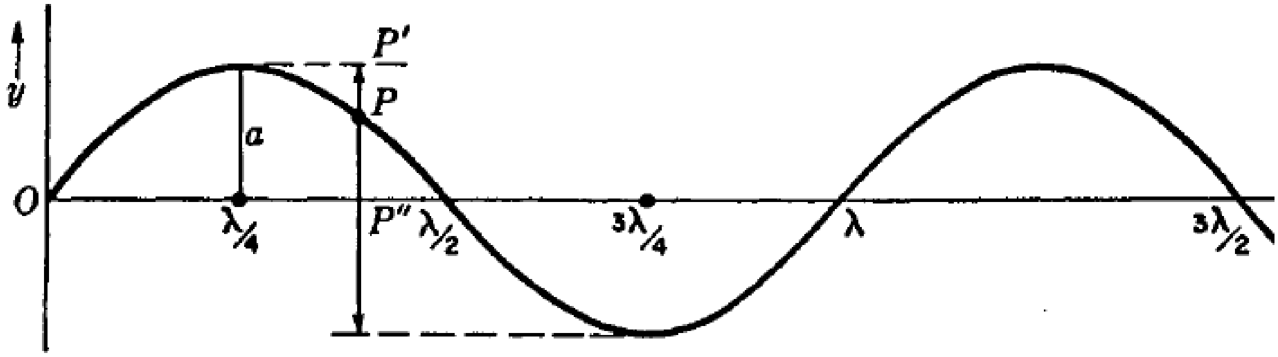

Fig. 5 Contour of a sine wave at time *t* = O.

Any particle of the wave, such as *P* in the diagram, will carry out SHM and will occupy successive positions *P*, *P'*, *P*, *P*''' etc., as the wave moves. The time for one complete vibration of anyone point is the same as any other point. Furthermore, the period *T* and its reciprocal the frequency υ are given by the wave equation (12):

$v = \upsilon\lambda = \frac{\lambda}{T}$	15

If we substitute several of these variables in Eq. (14), we can obtain useful equations for wave motion in general:

$y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$	16

$y = a\ sin \frac{2\pi}{T} \left(t - \frac{x}{v}\right)$	17

$y = a \sin 2\pi\nu \left(t - \frac{x}{v}\right)$	18

A useful and brief way of expressing the equation for simple harmonic waves is in terms of the *angular frequency* $w = 2\pi\upsilon$ and the *propagation number* $k = 2\pi/\lambda$. Equation (l4) then becomes:

$y = a\sin(kx - \omega t) = a\sin(\omega t - kx + \pi) = a\cos\left(\omega t - kx + \frac{\pi}{2}\right)$	19

The addition of a constant to the quantity in parentheses is of little physical significance, Thus the equations when written

$y = a\cos(\omega t - kx) \;\; and\; y = a\sin(\omega t - kx)$	20

will describe the wave of **Fig. 5**, if the curve applies at times *t* = *T/4* and *T/2*, respectively, instead of at *t* = 0.