Geometrical Optics can be expressed by a set of three laws:

1- The Law of Transmission:

In a region of constant refractive index, light travels in a straight line. The rectilinear propagation of light is obvious from the geometrical sharpness of the shadows cast by any light source e.g. sun, incandescent lamp.

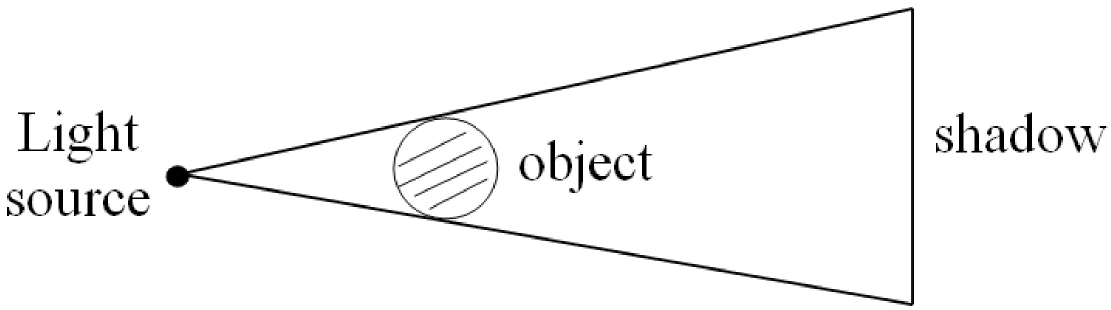


Fig.(1)

The property of the straight line propagation enables us to represent the behavior of beams of light by drawing straight lines which we called rays. This property is the result of the small wavelength.

2- Law of Reflection:

Light incident on a plane surface at an angle *i* with respect to the normal to the surface is reflected through an angle i' angle equal to the angle of incident *i*,

$$\theta_i = \theta_{i'}$$

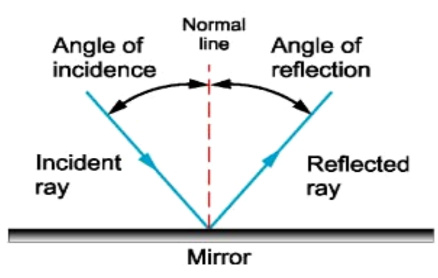


Fig.(2)

The reflected beam is in the plane determined by the incident beam and a line perpendicular to the reflecting surface.

3- Law of Refraction:

Light in medium 1 of refractive index n_1 incident on a plane surface AB at an angle i with respect to the normal is refracted to an angle r in medium 2 of refractive index n_2.

The refracted beam is in the plane determined by the incident beam and the perpendicular to the surface of separation

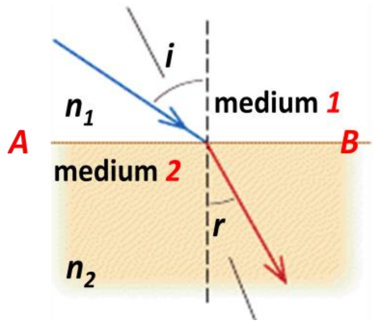


Fig.(3)

The ratio of the sine of angle of incident i to the sine of the angle of refraction r is a constant, known as the **Index of Refraction**.

In mathematical form:

$$\sin i / \sin r = {}_1n_2$$

The index of refraction for the passage of light from medium 1 to medium 2 is the index of the second medium relative to vacuum divided by the index of the first medium relative to vacuum, i.e.:

$$ {}_1n_2 = \frac{{}_an_2}{{}_an_1} = \frac{\sin i}{\sin r} \qquad \text{(a: air)}$$

$$\therefore \quad {}_an_1 \sin i = {}_an_2 \sin r \qquad \text{(Snell's Law)}$$

Remarks:

- From Snell's Law: $\quad {}_1n_2 = \frac{{}_an_2}{{}_an_1} = {}_1n_a \times {}_an_2$
- n depends on:

  1- The nature of the two media (e.g. their densities).

  2- The wavelength of light (usually given for yellow light).
- Light going from a less to a denser medium is refracted toward the perpendicular and vice versa.

- Light is refracted because it has different velocities in different media.
- The refractive index *n* is defined by the ratio of the velocity of light in vacuum to the velocity of light in the medium.

$$\boldsymbol{n = c/v} \qquad \mathbf{(c = 3 \times 10^{8}\ m.s^{-1})}$$

- The refractive index of air (or empty space, or vacuum) =1.
- Since n of other medium is > 1, thus the velocity of light in a medium is < c.

$$\boldsymbol{n} = \lambda_1/\lambda_2 = \frac{\lambda_1}{\frac{V_2}{f}} \quad \textbf{since} \quad \mathbf{V = f\,\lambda}$$

$$\boldsymbol{n} = \frac{\mathbf{f}\,\lambda_1}{\mathbf{V_2}} \qquad \mathbf{f = frequency,\ \lambda = wavelength}$$

Fermat's Principle:

Fermat's principle of least time states that "the path taken by a light ray is such that the time of travel is minimum". It means that the time is a minimum compared to that which would be occupied in traversing other possible paths close to the actual path. i. e.

$$\gamma \int_{t_1}^{t_2} dt = 0$$

t= time and the integral is taken between suitable limits. The velocity *V* of the wave is:

$$\boldsymbol{V = ds / dt}$$

The refractive index $\boldsymbol{n = C/V = C/\ ds/dt}$

Therefore, $\boldsymbol{dt = C\ n\ ds}$

By deleting *C*, the constant velocity of the light in vacuum, so that

$$\gamma \int n.ds = 0$$

$\int n.ds$ = optical path along the ray.

Optical Path:

The path of a ray of light in any medium equals to the product of the velocity and time, i. e

$$\boldsymbol{d = V\,t}$$

The refractive index of a medium is $\boldsymbol{n = C/V}$

$$V = C/n$$

So that, $$d = Ct/n$$

i.e $$nd = Ct = \Delta$$

The optical path $\Delta = nd$ = the distance light travel in vacuum in the same time it travels a distance d in the medium, i.e *nd* is the distance equivalent to the vacuum distance.

For a series of optical media of thickness d_1, d_2, d_3 … the total optical path Δ is:

$$\Delta = n_1 d_1 + n_2 d_2 + n_3 d_3 + \ldots$$

Critical Angle & Total Internal Reflection:

If a light ray travels from a dense medium to a less dense medium the emerging ray will be refracted at an angle greater than the angle of incidence.

There is a limiting angle of incidence called critical angle in the dense medium at which the light ray emerges along the surface, i.e. the angle of refraction in the less dense medium is 90^o.

$$n_1 > n_2$$

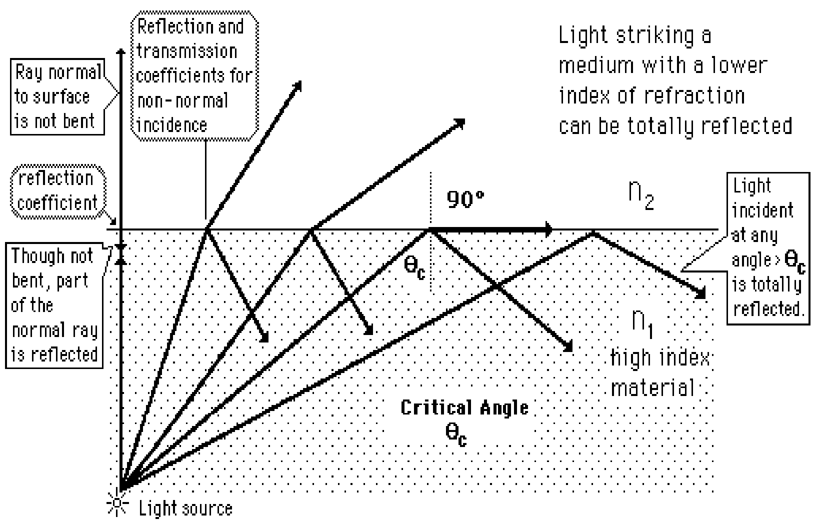


Fig.(4)

Applying Snell's law of refraction at the critical angle:

$$n_1 \sin \theta_c = n_1 \sin \theta_{90^o}$$

If the less dense medium is air, n_2=1 and hence

$$\boldsymbol{n_1 \ \sin\theta_c = 1} \quad \boldsymbol{, \sin\theta_c = \frac{1}{n_1}} < 1 \quad \boldsymbol{or} \quad \boldsymbol{n_1 = \frac{1}{\sin_{\theta_c}}} > 1$$

For water $\theta_c = 49^o$. If the angle of incidence is increased beyond the critical angle, the refracted ray disappears. The incident ray undergoes total internal reflection.

$$\boldsymbol{\theta_i > \theta_c}$$

Therefore, the critical angle is defined as "**the smallest angle of incidence in the medium of greater index beyond which light is totally reflected**".

The Principle of Reversibility:

If a reflected of or refracted ray is reversed in direction, it will retrace its original path.

Refraction at plane surfaces by paraxial rays:

Paraxial rays are defined as (rays for which angles are small enough to permit setting the cosines equal to unity and the sines and tangents equal to the angles in radians)

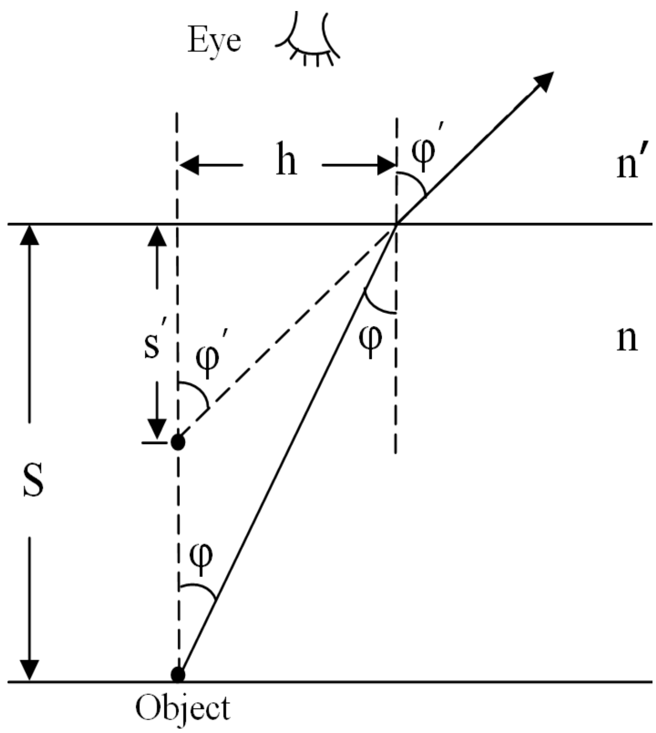


Fig. (5)

Assume $n > n'$

$$tan\,\emptyset = \frac{h}{s}$$

$$tan\,\emptyset' = \frac{h}{s'}$$

$$s\,tan\,\emptyset = s'\,tan\,\emptyset'$$

$$s\,\frac{sin\,\emptyset}{cos\,\emptyset} = s'\,\frac{sin\,\emptyset'}{cos\,\emptyset'}$$

From Snell's law: $n' \sin \emptyset' = n \sin \emptyset$

$$\therefore \quad \frac{s}{s'} \times \frac{n'}{n} = \frac{\cos \emptyset}{\cos \emptyset'} = 1,$$ since φ and φ' are very small

$$\therefore \quad \frac{s}{s'} = \frac{n}{n'}$$

$$i.e. \frac{Real\ depth}{Apparent\ depth} = \frac{n\ of\ medium}{n'\ of\ air}$$

Thus the ratio of the object to image distance for paraxial rays is equal to the ratio of the indices of refraction.

Reflection through a prism:

If a ray of light is sent through a prism and its deviation d is measured for various values of the angle of inclination i_1 in air at the first face of the prism, the graph of deviation d and angle of inclination i_1 takes the form shown Figure (6).

Experiment shows that as the angle of incidence i_1 is increased from zero, the deviation d beings to decrease continuously to some value D, and then increases to a maximum as i_1 is increased further to 90^o.

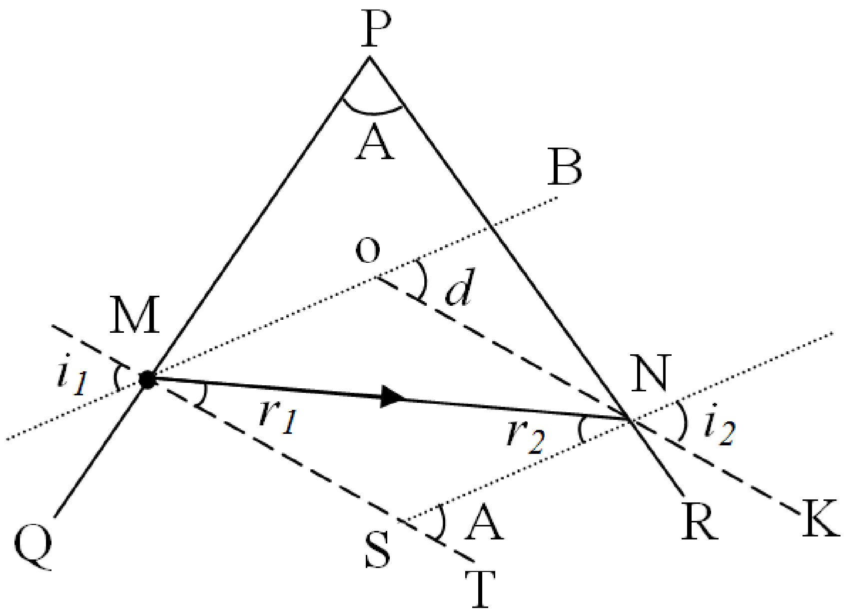

P
A
B
o
d
M
i_1
r_1
r_2
N
i_2
Q
S
A
T
R
K

Fig. (6)

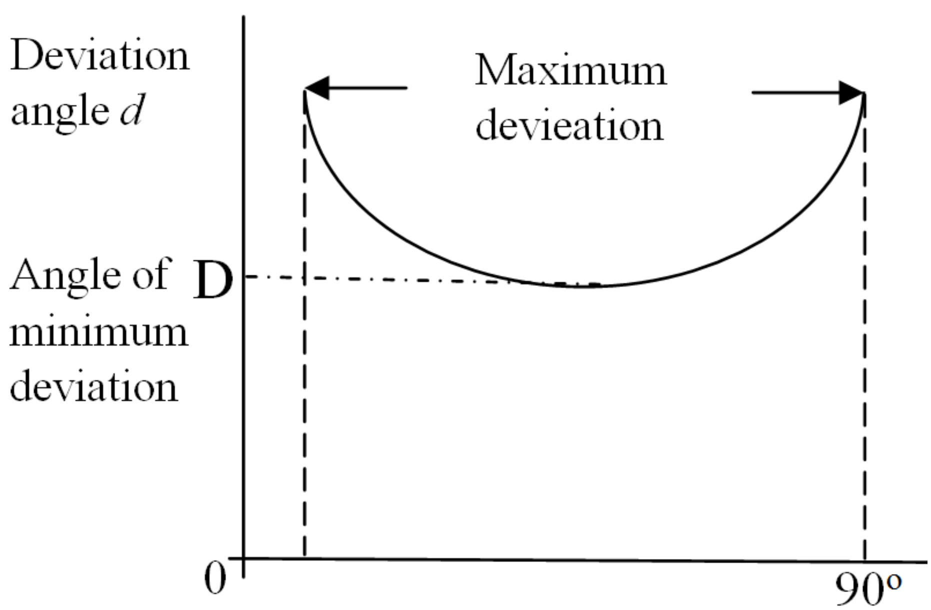

Deviation angle d
Maximum devieation
Angle of minimum deviation
D
0
90°

Fig. (7)

$sin\, i_1 = n\, sin\, r_1$

$\sin i_2 = n \sin r_2$

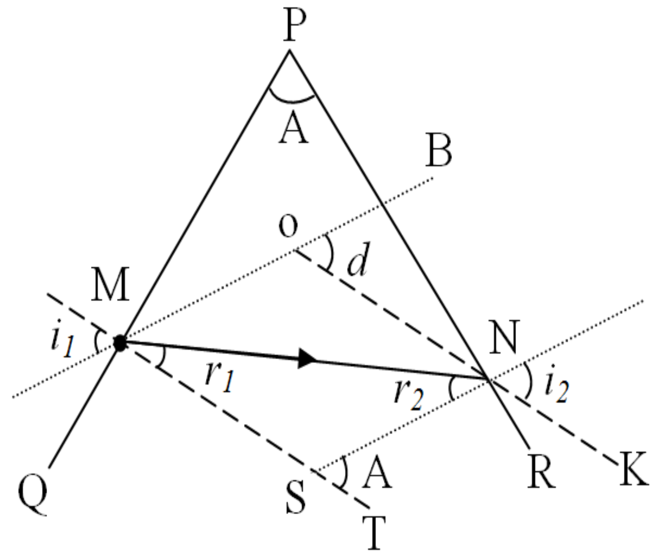


Fig.(6)

MS $\perp$ PM and NS $\perp$ PN

Δ MPN + Δ MSN = 180^o

But Δ NST + Δ MSN = 180^o

∴ Δ NST = Δ MNP = A

∴ **A = $r_1 + r_2$**

The angle of deviation d = Δ BOK

= Δ OMN + MNO

$$d = (i_1 - r_1) + (i_2 + r_2)$$

There is one angle of inclination for which the deviation is a minimum. The ray of light at minimum deviation passes through the prism symmetrically, so that the angle at which it emerges from the prism is equal to the angle at which it enters. Hence $i_1 = i_2 = i$ and $r_1 = r_2 = r$

Angle of minimum deviation D = Δ LFG + Δ LGF = $(i - r) + (i - r)$

∴ D = $2i - 2r$ … (1)

A = $r_1 + r_2 = 2r$

i.e. r = A / 2

From eq. (1)

$i = \frac{D+2r}{2} = \frac{D+A}{2}$

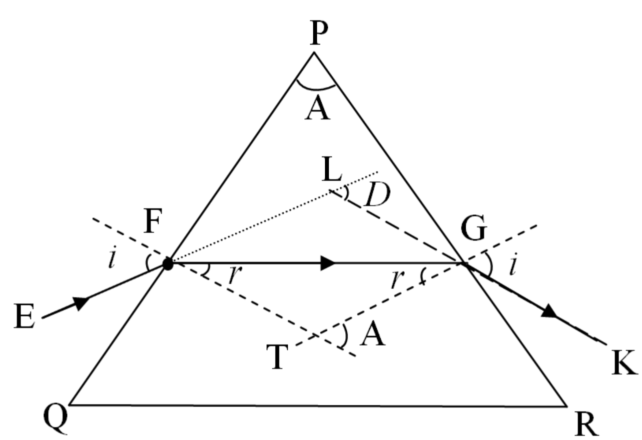


Fig. (8)

From Snell's law: $\boldsymbol{n_1 \sin i = n_2 \sin r}$

$n_1 = 1$ for air and $n_2 = n$ for glass relative to air

$$❖ \quad n = \frac{\sin i}{\sin r} = \frac{\sin\left(\frac{D+A}{2}\right)}{\sin\frac{A}{2}}$$

Therefore, from the angle of prism and angle of minimum deviation the refractive index of the material of the prism can be determined for a specific color. The **spectrometer** is an instrument designed for this purpose.

Color Dispersion:

The refractive index is not a constant but varies as a function of wavelength λ. The presence of this variation is called **dispersion** (*dn/dλ*) of the material. Thus refraction causes a separation of white light into its component color, i.e. white light is spread out into a spectrum. The symbols F, D, and C are the names of the Fraunhofer wavelength in the solar spectrum (dark lines).

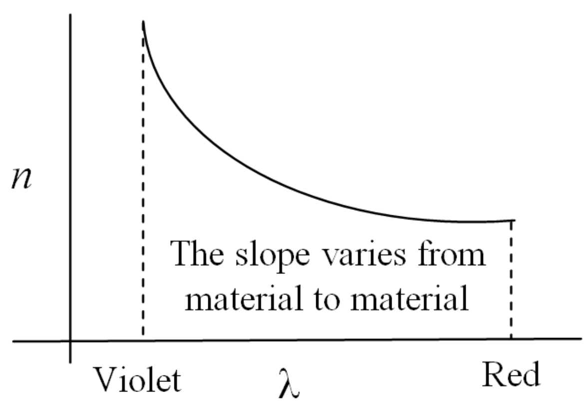


Fig. (9)

A number that indicates the amount of the dispersion (or dispersing properties) for a glass is the V-number, or Abbe′ number, defined by:

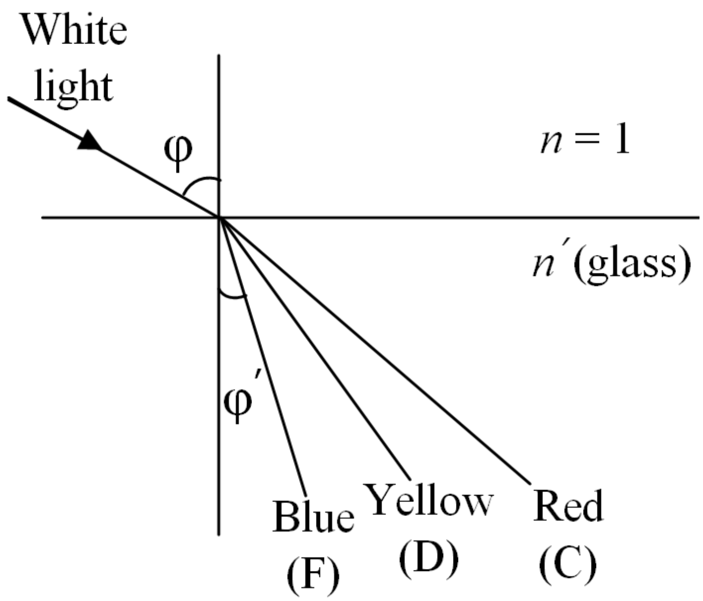


Fig. (10)

$$V = \frac{n_D - 1}{n_F - n_C} = dispersive\ index$$

$20 \leq V \leq 60$ for glass

The dispersive power = $\frac{1}{V} = \frac{n_F - n_C}{n_{D-1}}$

A large value of *V* indicates small dispersion. Because the dispersion (i.e. dispersive power) is greatest when (n_F – n_C) is largest, glasses with the strongest dispersion have the smallest *V*-number.

The angular dispersion ($\emptyset'_C - \emptyset'_F$) is proportional to ($n_F$ – n_C). the deviation of the yellow (D) ray is ($\emptyset - \emptyset'_D$) which is proportional to (n_D – 1).

While many glasses have special names (crown glass K, flint glass F) all glasses can be labeled by their n_D and *V*. the glass number is a six digits number whose first three digits are the three most significant digits of n_D – 1 and whose second three digits are ten times (10 ×) the *V*-number. For example, the glass number for crown glass (n_D = 1.5320 and V = 58.7) is 532587.

Flint glass has a greater dispersion than crown since n of the flint glass › n of crown glass.

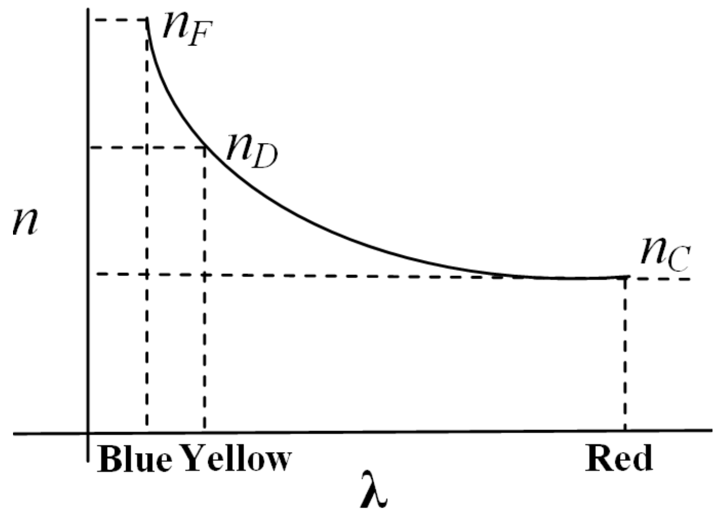


Fig. (11)

Remarks:

The first successful attempt to represent the curve of normal dispersion by an equation was made by Cauchy in 1836. This equation is:

$$n = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}$$

Where A, B, and C are constants which are characteristics of any one substance. At sufficient approximation $n = A + \frac{B}{\lambda^2}$. Hence, the dispersion becomes $\frac{dn}{d\lambda} = \frac{-2B}{\lambda^3}$. The mines sign corresponds to the usual negative slope of the n vs. λ curve. The resolving power $\frac{\lambda}{d\lambda}$ of a prism can be shown to be $t.\frac{dn}{d\lambda}$ (i.e. $\frac{\lambda}{d\lambda} = \frac{-2B}{\lambda^3}.t$) where t is the difference in path travelled by the extreme rays through the prism. If the whole of the prism is used, t is the length of the base. The angular dispersion of prism $\frac{d\emptyset}{d\lambda} \times \frac{dn}{d\lambda}$ and $\frac{\lambda}{d\lambda} \times \frac{dn}{d\lambda}$.

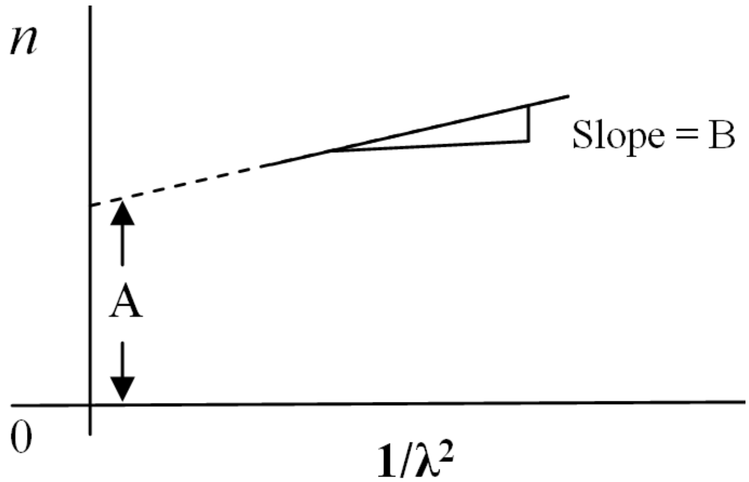


Fig. (12)

Remarks

Ray AO is incident at O on a plane mirror M_1. If OB is the reflected ray, the angle of deviation COB = 2α.

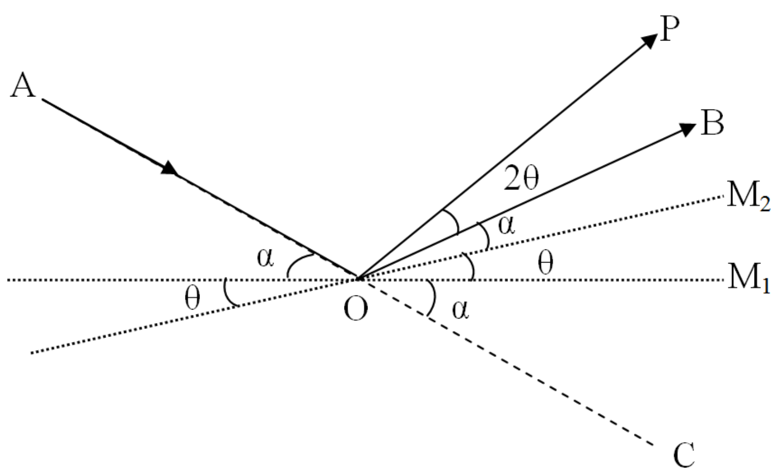


Fig. (13)

If the mirror is rotated through an angle θ to a position M_2, the direction of the incident ray AO being constant, then the ray is now reflected from M_2, in a direction OP.

The angle of incident = $\alpha + \theta$

The new angle of deviation COP = $2(\alpha + \theta)$

The reflected ray has thus been rotated through an angle BOP when the mirror rotated through an angle θ

Since $\sphericalangle BOP = \sphericalangle COP - \sphericalangle COB$

$\therefore \quad \sphericalangle BOP = 2(\alpha + \theta) - 2\alpha$

$= 2\theta$

Thus if the direction of an incident ray is constant, the angle of rotation of the reflected ray is twice the angle of rotation of the mirror.

Reflection of Light at Curved Mirrors:

Curved mirrors are parts of spherical surfaces. There are two types of curved mirrors:

1- Concave (converging) Mirrors:

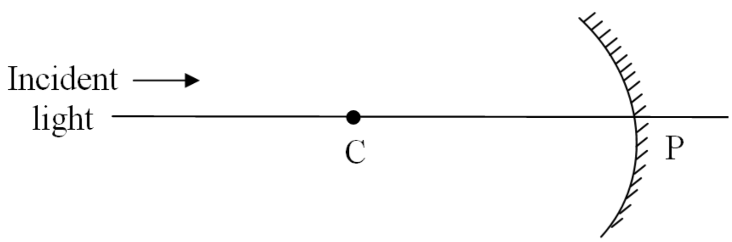


Fig. (14)

The center C is in front of the reflecting surface.

2- Convex (diverging) Mirrors:

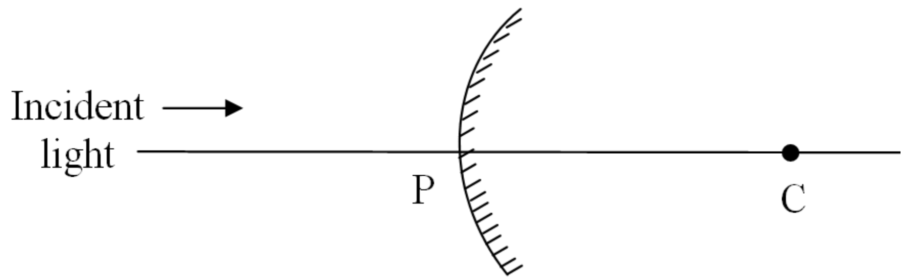


Fig. (15)

The center C is behind the reflecting surface.

C: The center of the curvature.
P: The pole of the mirror.
PC: The principle axis.

Focal Length (f) and Radius of Curvature (r):

Consider a ray AX parallel to the principle axis. The normal to the mirror at X is CX because the radius of the spherical surface is perpendicular to the surface and hence the reflected ray makes an angle θ with CX equal to the incident angle θ.

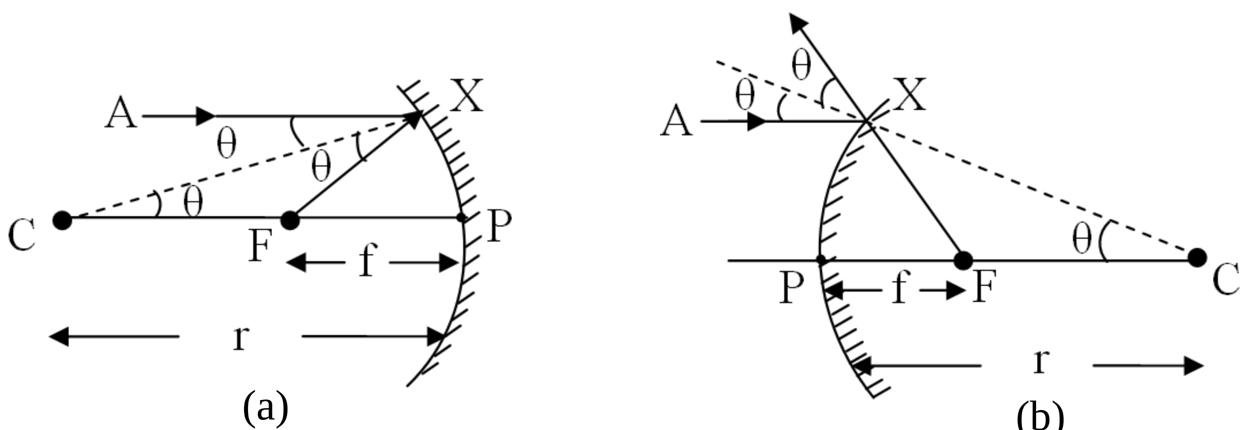


Fig. (16)

PC = radius of curvature (r), PF = focal length (f).

For the case of the concave mirror, the alternative angles are equal:

$$\angle AXC = \angle XCP$$

$\therefore \quad \Delta$ FXC is isosceles and XF = CF.

As X is a point very close to P, we assume to a very good approximation that FX = FP.

$\therefore$ FP = CF or FP = ½ CP

$$\therefore \quad f = \frac{r}{2}$$

This relation between f and r is the same for the case of the convex mirror.

$\therefore$ spherical mirror have <u>one</u> focal point.

Images in Concave Mirror:

Concave mirrors produce images of different sizes; sometimes they are inverted and real and on other cases they are erect and virtual. The image is located by the intersection of two reflected rays coming from H by choosing two of the following:

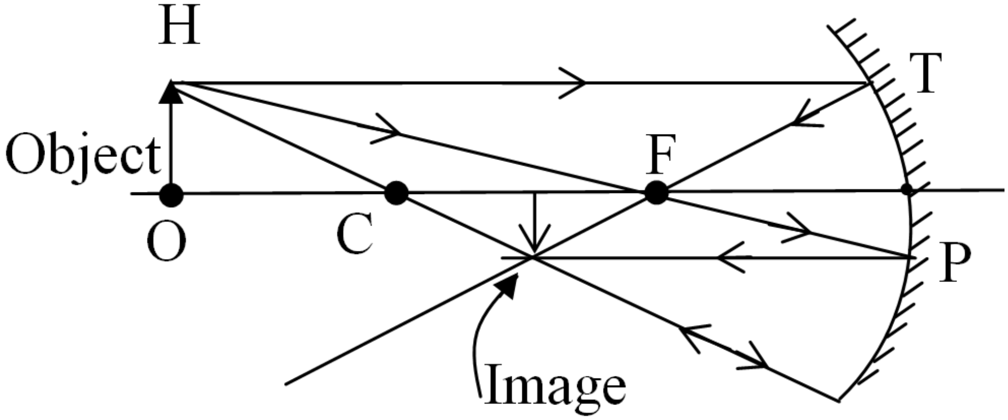


Fig. (17)

(a) – Ray HT parallel to the principle axis which is reflected to pass through focus F.

(b) – Ray HC passing through the center of curvature C which reflected back along its own path because it is a normal to the mirror.

(c) – Ray HF passing through the focus F which is reflected parallel to the principle axis.

When the object is at infinity, the image is small and is formed inverted at the focus.

Images in Convex Mirrors:

Ray HM parallel to the principle axis is reflected as it appeared to come from the virtual focus F. ray HN incident towards the center of curvature C is reflected back along its path.

The two reflected rays intersect behind the mirror at R and IR is a vertical and erect image.

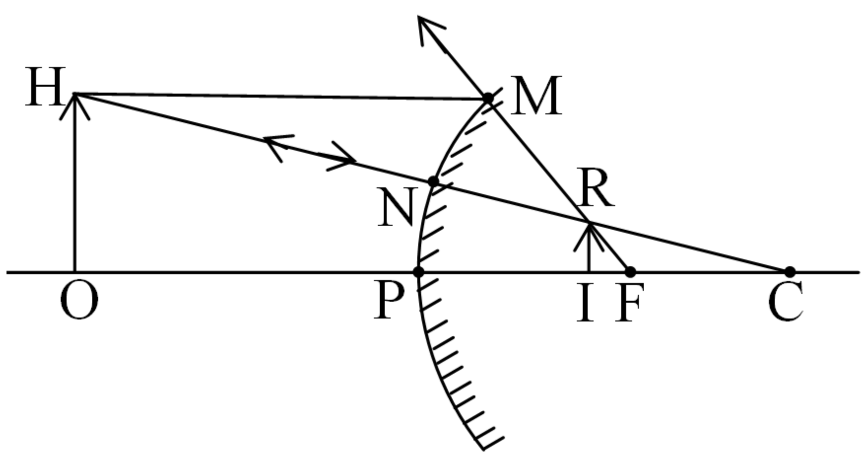


Fig. (18)

Sign convention:

(a) – Incident light would always be shown coming from the left-hand side.

(b) – Pole of the mirror would taken as the origin for the purpose of measuring various distance.

(c) – The distances of the real objects or images from the mirror would be taken a positive but those of virtual images would be regarded as negative.
i.e. Real positive, Virtual negative.

The distance actually travelled by light are taken as positive, those not actually travelled as negative.

Don't give any sign to any unknown quantity in anticipation. The result will itself indicate its actual sign.

Concave mirrors will always have positive focal length f and positive radius of curvature r.

Convex mirrors will always have f and r negative.

Concave Mirror:

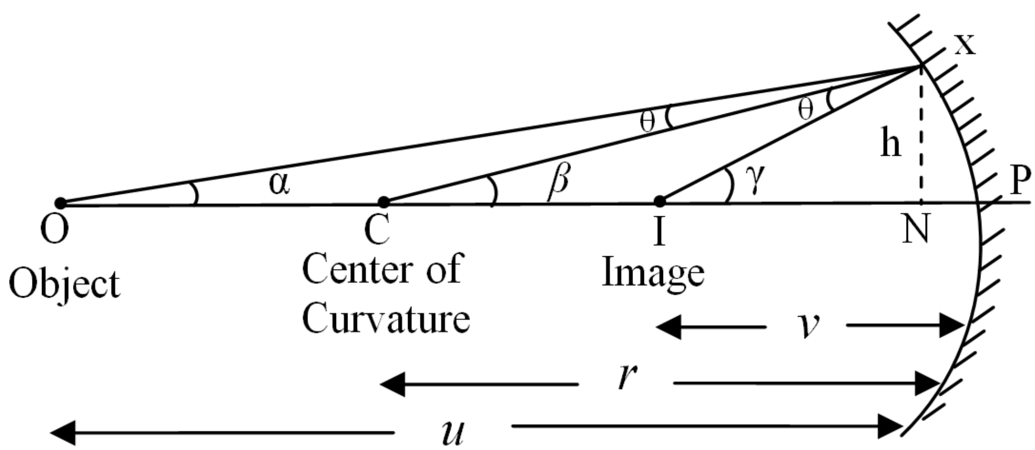


Fig. (19)

Since we are considering a mirror of small aperture, angles α, β, and γ are very small. β is the exterior angle of Δ CXO,

$$\beta = \alpha + \theta$$

$$\therefore \theta = \beta - \alpha \text{ .... (1)}$$

γ is the exterior angle of Δ IXC,

$$\gamma = \beta + \theta$$

$$\therefore \theta = \gamma - \beta \text{ .... (2)}$$

From eq. (1) and (2): β – α = γ – β

$$\therefore \alpha + \gamma = 2\beta \text{ .... (3)}$$

N is practically coincident with P.

$$\tan \alpha = \alpha \text{ in radians}$$

$$\therefore \alpha = \frac{XN}{ON} = \frac{h}{+OP}$$

$$\tan \beta = \beta \text{ in radians}$$

$$\therefore \beta = \frac{XN}{CN} = \frac{h}{+CP}$$

$$\tan \gamma = \gamma \text{ in radians}$$

$$\therefore \gamma = \frac{XN}{IN} = \frac{h}{+IP}$$

Equation (3) becomes:

$$\frac{h}{(+OP)} + \frac{h}{(+IP)} = 2\frac{h}{(+CP)}$$

i.e. $\frac{1}{OP} + \frac{1}{IP} = \frac{2}{CP}$

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r}$$

$$\frac{2}{r} + \frac{1}{2f} = \frac{1}{f}$$

$$\left[\therefore \frac{1}{u} + \frac{1}{v} = \frac{1}{f}\right]$$

Convex Mirror:

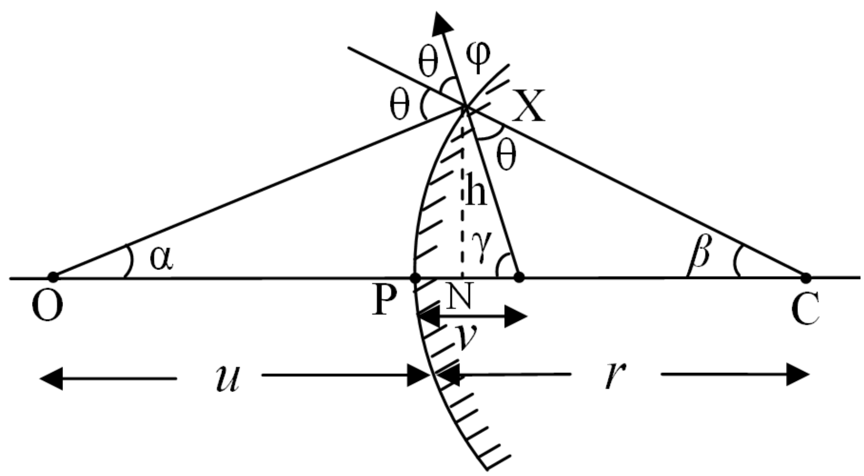


Fig. (20)

Angles α, β, γ, and θ are very small.

θ is the interior angle of Δ COX

$$\theta = \alpha + \beta \ \ldots\ldots. (1)$$

γ is interior angle of Δ CIX

$$\gamma = \theta + \beta \text{ or } \theta = \gamma - \beta \ \ldots\ldots. (2)$$

Therefore, from equation (1) and (2):

$$\gamma - \beta = \alpha + \beta$$

$$\gamma - \alpha = 2\beta \text{ .... (3)}$$

$\gamma = \frac{h}{IN} = \frac{h}{(-IP)}$ since I is virtual

$\alpha = \frac{h}{ON} = \frac{h}{(+OP)}$ since O is real

$\beta = \frac{h}{NC} = \frac{h}{(-PC)}$ since C is virtual

By substituting in equation (3):

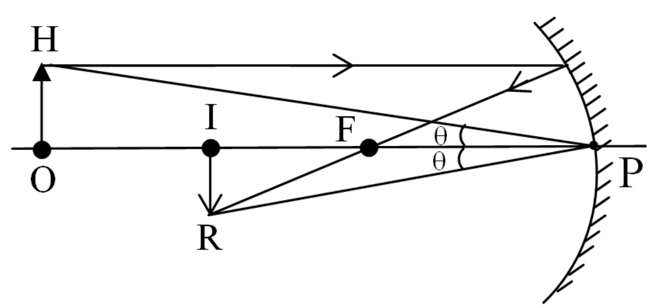
$$\frac{h}{(-IP)} - \frac{h}{(+OP)} = \frac{2h}{(-CP)}$$

$$\frac{1}{IP} + \frac{1}{OP} = \frac{2}{CP}$$

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$$

$$\left[\therefore \frac{1}{u} + \frac{1}{v} = \frac{1}{f}\right]$$

Formula of Magnification:



$$Magnification\ m = \frac{Height\ of\ image}{Height\ of\ object}$$

From the law of reflection:

$\sphericalangle$ OPH = $\sphericalangle$ IPR = θ

$\therefore \tan OPH = \tan IPR$

$$\frac{OH}{OP} = \frac{IR}{IP}$$

$$\therefore \frac{IR}{OH} = \frac{IP}{OP}$$

i.e. $\frac{Image\ Height}{Object\ \ Height} = \frac{Image\ distance\ (v)}{Object\ distance\ (u)}$

$$\therefore \left[m = \frac{v}{u}\right]$$

Method of Determining *f* and *r* of spherical mirrors:

A convex lens forms the image of an object O at point C. The convex mirror MN is then placed between the lens and the image. When the mirror is moved along the axis OC, a position is reached when the converging beam of light falls normally on it. In that case, the rays of light are reflected back along the incident path and forms a real inverted image at O.

PC = radius of curvature , *r* = LC – LP = *2f*

LC = image distance from the lens when the mirror is taken away.

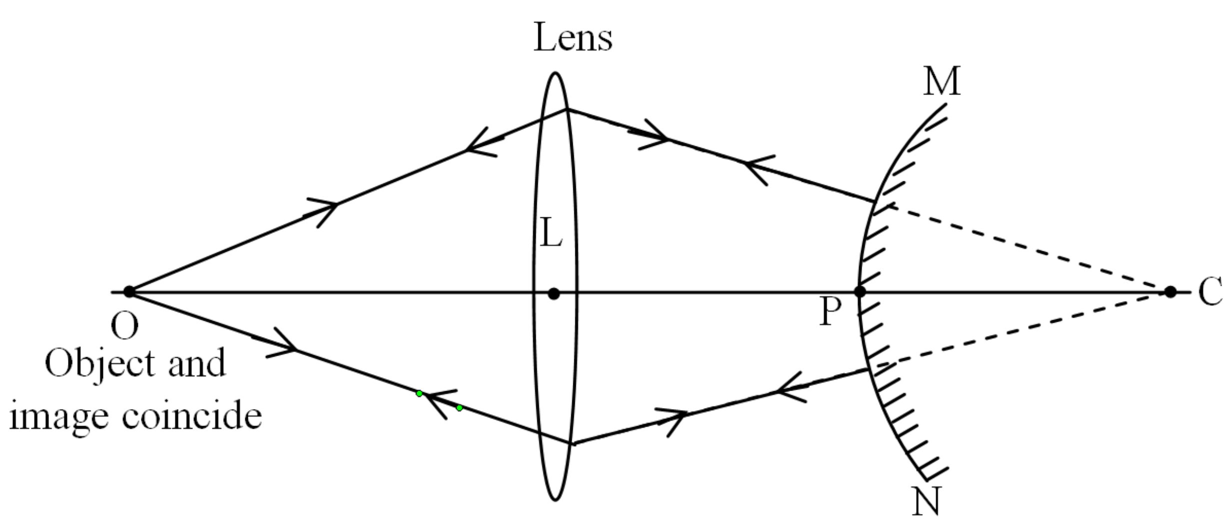


Fig. (21)

Lenses:

A lens is a piece of reflecting material bounded by two surfaces having a common axis. A lens is said to be thin if its thickness is very small as compare to the distance of the objects and images from it. Lenses may be convex or concave. A convex lens is a converging lens since a parallel beam of light after refraction converges to a point F called the principal focus. A concave lens is a diverging lens since rays coming parallel to the principal axis after refraction through the lens diverge out and seen to come from a point like F called the principal focus.

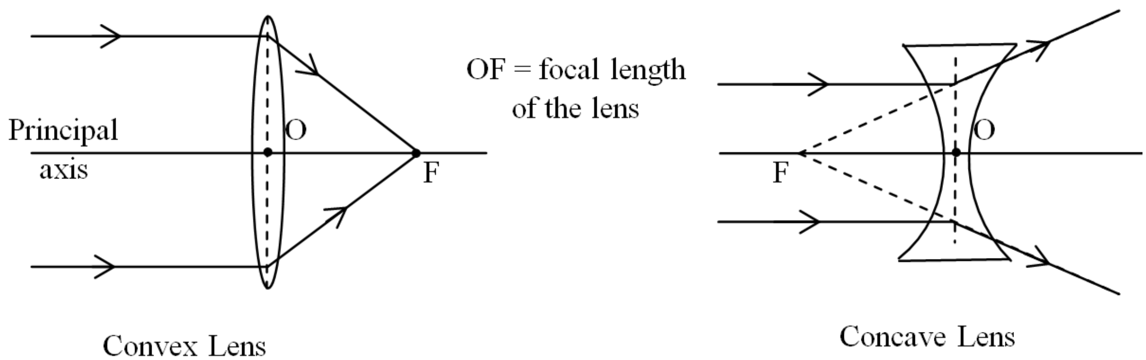


Fig. (22)

Remark:

The ray AB passing through the optical center O will go un-deviated.

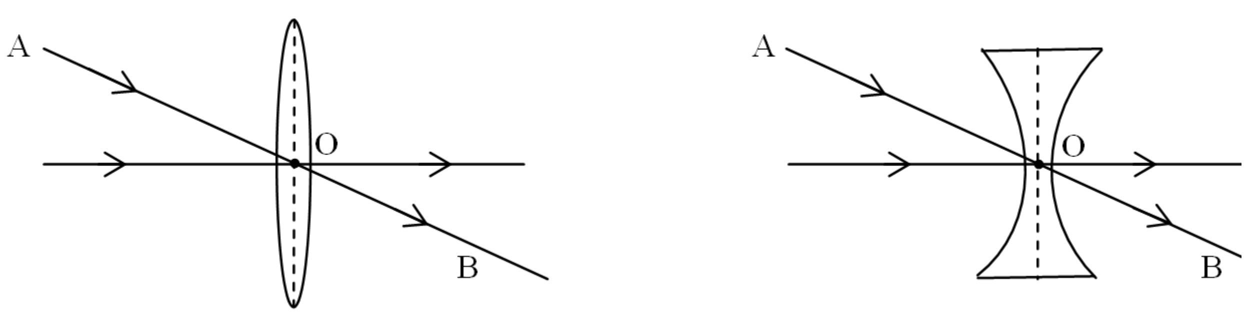


Fig. (23)

The power of a lens is a reciprocal of its focal length in meters. The unit of power is **<u>diopter</u>**. Doipter is defined as the power of a lens of focal length 1 meter. If *f* is the focal length of a lens in cm, then its power is $p = \frac{1}{f/100}$ diopters.

Forms of a Convex Lens:

1- Double- convex or bi-convex: if R_1 and R_2 are equal, it is called equi-convex lens.

Fig. (24)

2- Plano-convex lens:

Fig. (25)

3- Concavo-convex lens:

Fig. (26)

Forms of a Convex Lens:

1- Double- concave or bi-concave: if R_1 and R_2 are equal, it is called equi-concave lens.

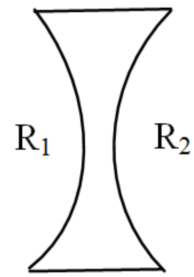


Fig. (27)

2- Plano-concave lens:

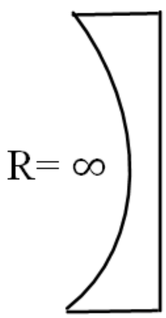


Fig. (28)

3- Convexo-concave lens:

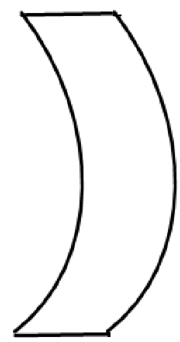


Fig. (29)

Images Formed by a Convex Lens:

1- Object at infinity: the image is at I, $u = \infty$, $v = f$.

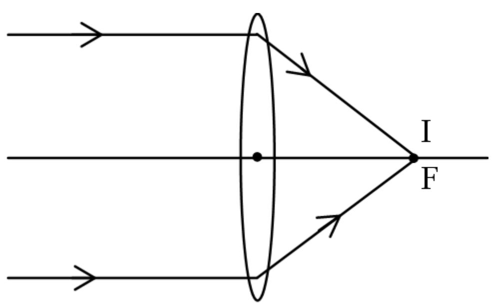


Fig. (30)

2- Object between infinity and center of curvature C: image is between C and F; it **real**, **inverted**, and **smaller** than the object.

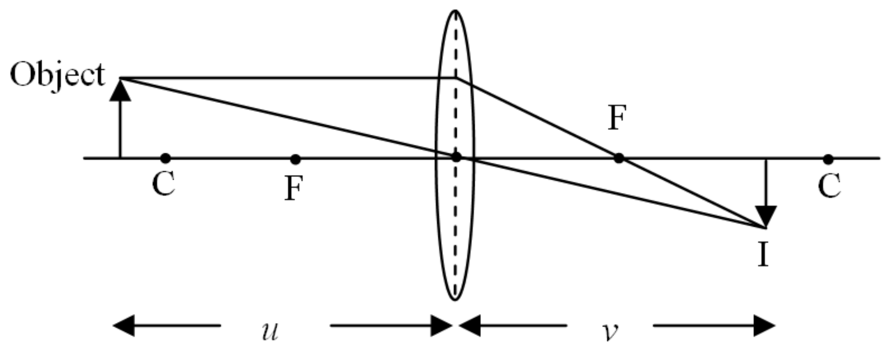


Fig. (31)

3- Objet at C: image is at C also; it is **real**, **inverted**, and of the **same size** as the object.

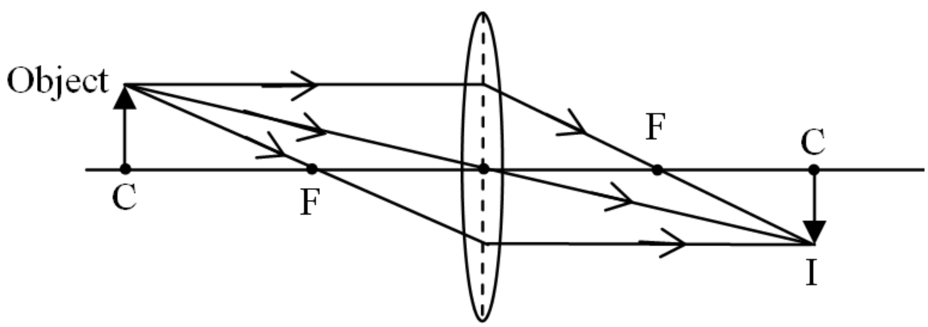


Fig. (32)

4- Object between C and F: image is beyond C; it is **real**, **inverted**, and **larger** than the object.

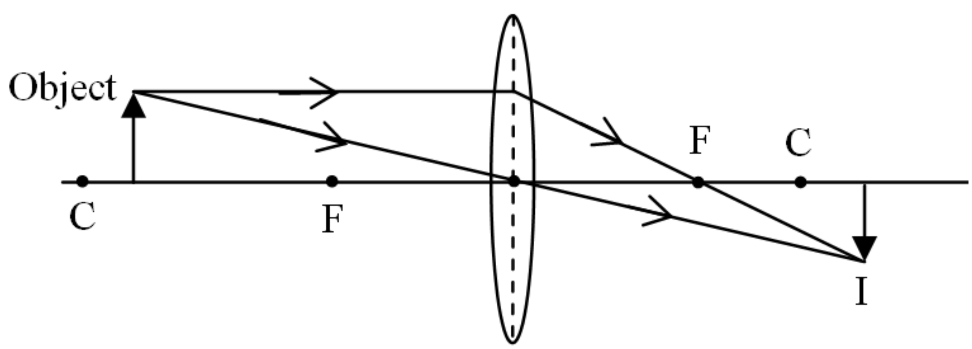


Fig. (33)

5- Object at F: image is at infinity.

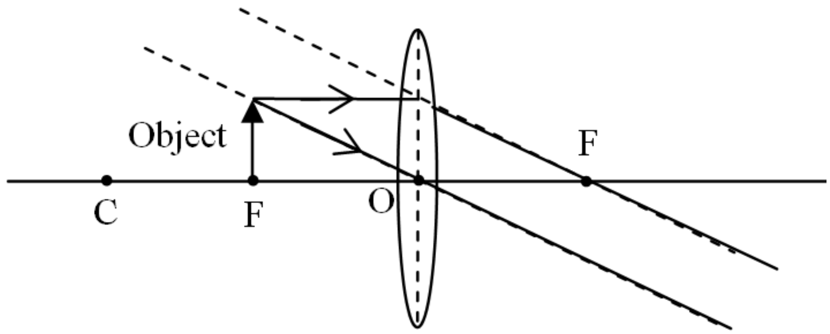


Fig. (34)

6- Object between F and the optical center O: the image is **virtual**, **erect**, and **magnified**. This case represents the **simple microscopic**.

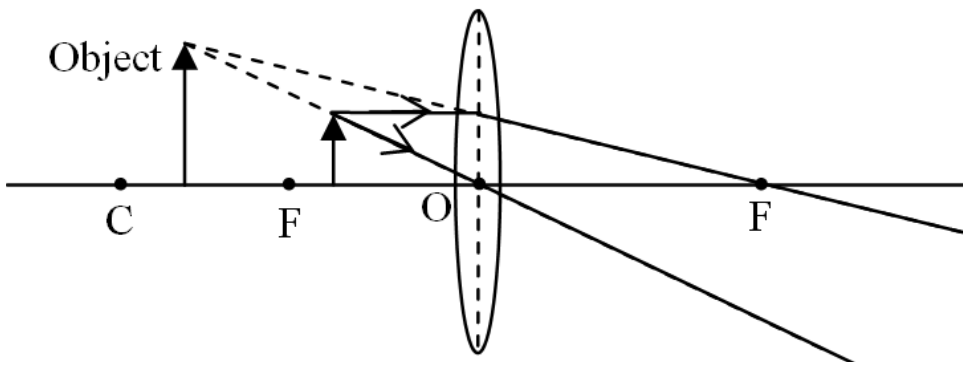


Fig. (35)

Images Formed by a Concave Lens:

A concave lens always forms virtual images irrespective of the distance of the object from it. The image always lies between F and O; it is virtual and smaller than the object.

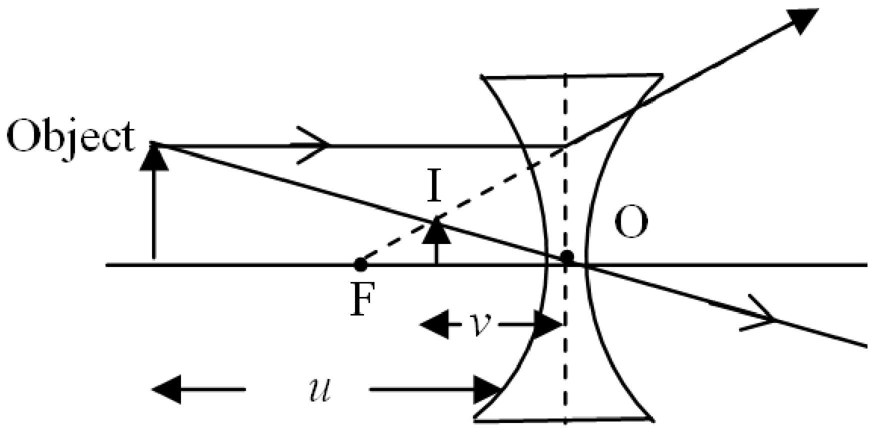


Fig. (36)

Simple Lens Formula:

(a) Convex Lens:

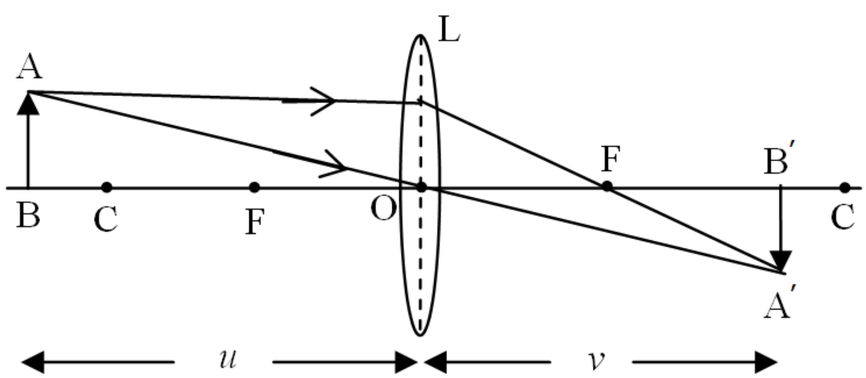


Fig. (37)

Δ AOB and Δ A′OB′ are similar,

therefore $\frac{AB}{A'B'} = \frac{OB}{O'B'}$ …. (1)

Since AB = OL, hence $\frac{AL}{A'B'} = \frac{OB}{O'B'}$ …. (1)

Also Δ OFL and Δ A′FB′ are similar.

$$\therefore \frac{OL}{A'B'} = \frac{OF}{FB'} \quad ….. (2)$$

From equation (1) and (2)

$$\frac{OB}{OB'} = \frac{OF}{FB'}$$

Or

$$\frac{OB}{OB'} = \frac{OF}{OB' - OF}$$

Or

$$\frac{u}{v} = \frac{f}{v - f}$$

(*u* and *v* are positive)

Cross- multiplying and simplifying we get

$$uf + vf = uv \quad \text{.....} \; (3)$$

Dividing both sides of equation (3) by *u v f*, we get:

$$\left[\frac{1}{u} + \frac{1}{v} = \frac{1}{f}\right]$$

(b) Concave Lens:

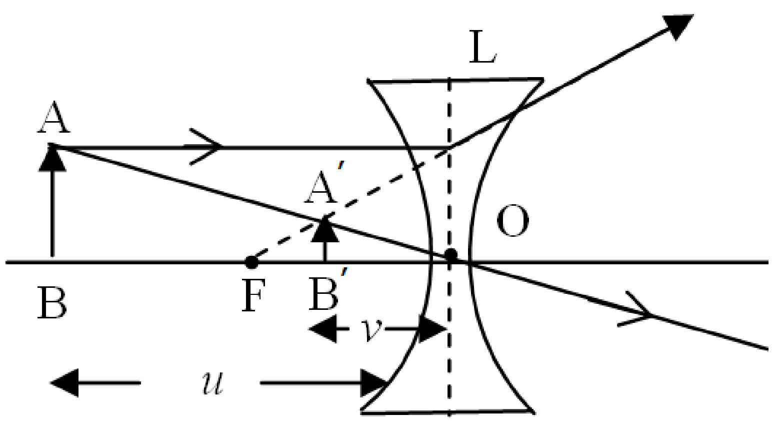


Fig. (38)

Δ AOB and Δ A′OB′ are similar, therefore $\frac{AB}{A'B'} = \frac{OB}{OB'}$

Since AB=OL, hence $\frac{OL}{A'B'} = \frac{OB}{OB'}$ ….. (1)

Also Δ. OFL and A′FB′ are similar.

$$\frac{OL}{A'B'} = \frac{OF}{FB'} \quad …..(2)$$

From equation (1) and (2)

$$\frac{OB}{OB'} = \frac{OF}{FB'}$$

or $\frac{OB}{OB'} = \frac{OF}{OF-OB'}$

or $\frac{u}{v} = \frac{f}{f-v}$

According to the sign convention, *u* is positive , *v* and *f* both are negative. Hence, $\frac{u}{-v} = \frac{-f}{-f-(-v)}$

or $uf + vf = uv$ ….. (3)

Dividing both sides of equation (3) by *uvf*, we get $\left[\frac{1}{u} + \frac{1}{v} = \frac{1}{f}\right]$

Conjugate Points and Newton's Relation:

Consider the following figure:

Since light rays ar reversible, hence an object at I will give rise to an image at O. the points O and I are interchangeable and hence are called conjugate points with respect to the lens. Newton showed that the conjugate points obey the relation $xx' = f^2$.

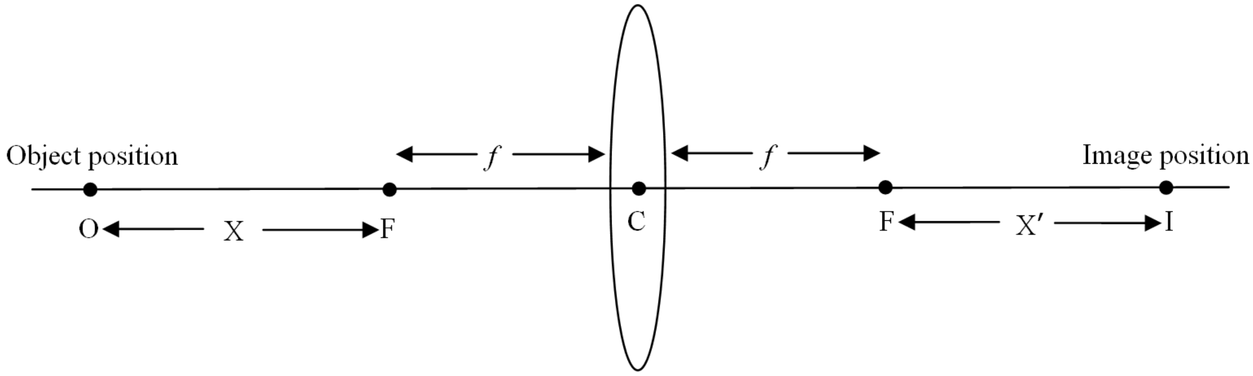


Fig. (39)

Proof:

Object distance OC = u = X+ f

Image distance CI = v + X′ + f

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$$

$$\frac{1}{(X+f)}+\frac{1}{(X'+f)}=\frac{1}{f}$$

$$f\,(x'+f)+f\,(X+f)=(X'+f)\,(X+f)$$

$$f\,(X'+X+2f)=XX'+fX'+fX\,f^2$$

$$[\therefore XX'=f^2]$$

i.e $X'\alpha\,\frac{1}{X}$, thus when X decreases, X′ increases.

Least Possible Distance Between Object and Real Image with Converging Lens:

Theory: The distance between an object and a screen must be equal to, or greater than, four times the focal length if a real image is required.

Proof: Consider the following figure:

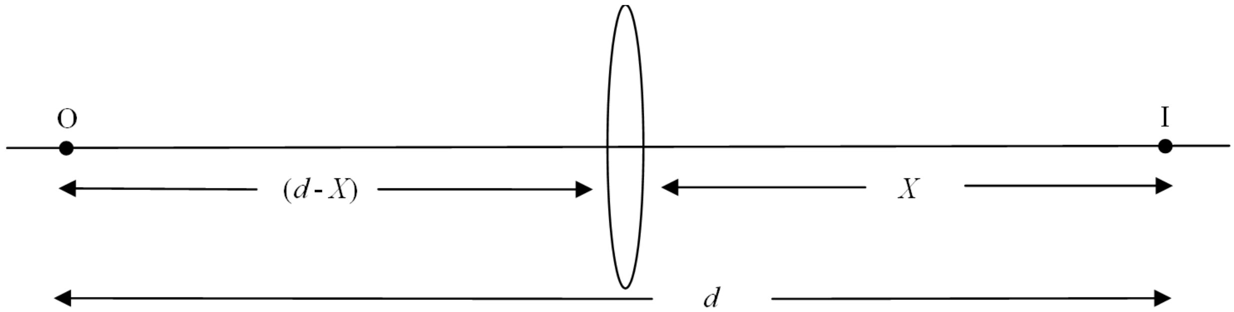


Fig. (40)

I is a real image of a point object O. therefore, $v = +X$ and $u = +(d - X)$

$$\therefore \frac{1}{X} + \frac{1}{d-X} = \frac{1}{f} \quad \text{i.e.} \quad \frac{d}{X(d-X)} = \frac{1}{f}$$

or $X^2 - dX + df = 0$; this is a quadratic equation of the general form $aX^2 + bX + c = 0$. For a real image the roots of this quadratic equation for X must be real roots, i.e. $b^2 - 4ac \geq 0$.

$$\therefore d^2 - 4df \geq 0 \text{ or } d^2 \geq 4df$$

$$[\therefore d \geq 4f]$$

Hence, the minimum distance between an object and its real image for a convex lens must be 4*f* where *f* is the focal length of the lens.

Refractive Index of a Liquid by Using a Concave Mirror:

When image and object coincide, the rays from the pin must strike the mirror normally. Therefore, the rays are reflected back along the incident path, i.e. ray ND strikes the mirror normally and when produced, it passes through the center of curvature C.

$\sphericalangle$ ANH = $\sphericalangle$ NHM = i

$\sphericalangle$ BND = $\sphericalangle$ ANC = r

From Δ HNM and Δ CNM

$\sin i = \frac{NM}{HN}$ and $\sin r = \frac{NM}{CN}$

The refractive index of the liquid is n.

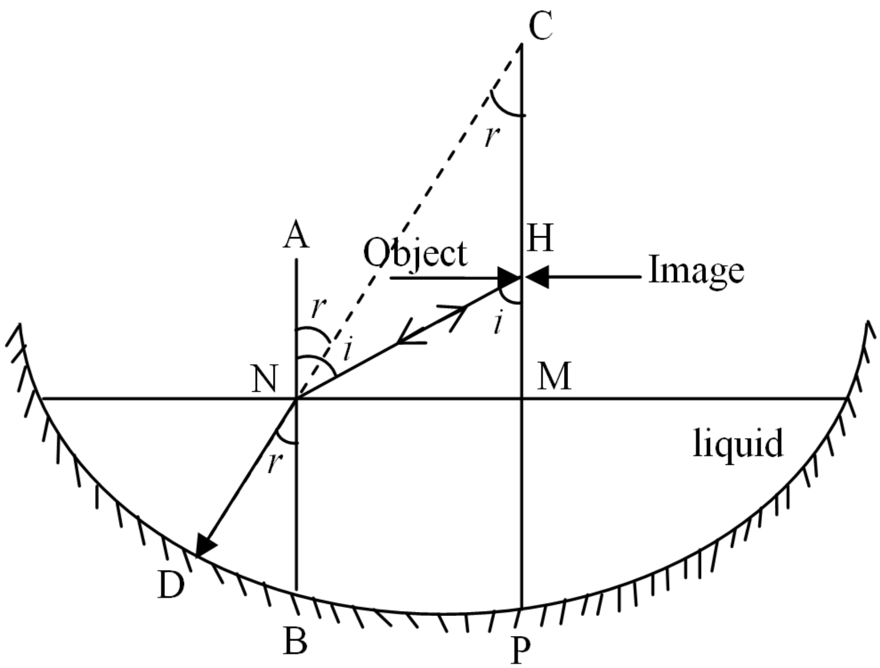


Fig. (41)

$n = \frac{\sin i}{\sin r}$ according to Snell's law.

$= \frac{NM/HN}{NM/CN} = \frac{CN}{HN}$

The ray HN is very close to the principal axis CP, thus

$$HN = HM \text{ and } CN = CM$$

$$\therefore n = \frac{CM}{HM}$$

If the depth MP of the liquid is very small compared with HM and CM, thus

$$CM = CP \text{ and } HM = HP$$

$$\therefore n = \frac{CP}{HP} = \frac{R}{HP}$$

where R is the radius of curvature of the concave mirror.

$$R > HP$$

Hence $n > 1$.

Refraction at Curved surfaces:

Theorem: Any spherical refracting surface forms a point image on its axis of a point object on the axis if only paraxial rays can pass through it.

Consider the following figure. It is assumed that the refractive index $n_2 > n_1$

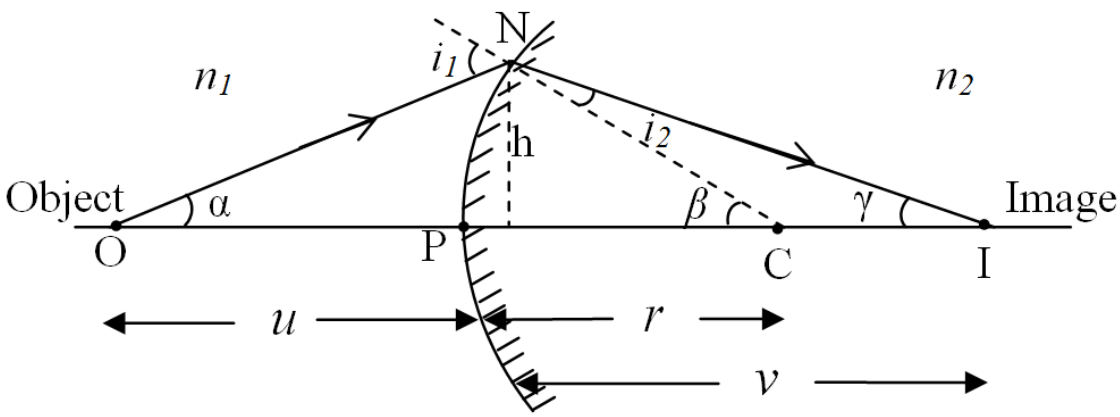


Fig. (42)

Ray ON is very close to the optical axis, i.e. ON is a paraxial ray. According to Snell's law:

$n_1 \sin i_1 = n_2 \sin i_2$ …. (1)

since i_1 is small for paraxial rays, hence

$\sin i_1 = i_1$ in radians

and similarly $\sin i_2 = i_2$ in radians

$$\therefore n_1 i_1 = n_2 i_2 \quad …. (2)$$

From Δ ONC : $i_1 = \alpha + \beta$

From Δ CNI : $i_2 = \beta - \gamma$

Equation (2) becomes:

$$n_1\,(\alpha + \beta) = n_2\,(\beta - \gamma)$$

$$\text{i.e. } n_1\,\alpha + n_2\,\gamma = (n_2 - n_1)\,\beta \;\ldots\; (3)$$

$$\alpha = \frac{h}{OP}, \qquad \gamma = \frac{h}{PI}, \qquad \beta = \frac{h}{PC}$$

Since real is positive, therefore,

$$h\left(\frac{n_1}{+OP} + \frac{n_2}{+PI}\right) = h\left(\frac{n_2 - n_1}{PC}\right)$$

$$\therefore \frac{n_1}{OP} + \frac{n_2}{PI} = \frac{n_2 - n_1}{PC}$$

$$\text{i.e. } \left[\frac{n_1}{u} + \frac{n_2}{v} = \frac{n_2 - n_1}{r}\right] \ldots\; (4)$$

It is to be noted that u corresponds to the refractive index n_1 of the medium in which the object is situated, while v corresponds to the medium of refractive index n_2 in which the image is situated. Equation (4) is Gaussian formula for a single spherical surface. The power of the surface is given by:

$$\frac{(n_2 - n_1)}{r}$$

Since n_1 and n_2 have no sign thus $(n_2 - n_1)$ is taken always as a positive quantity, i.e. $|n_2 - n_1|$. If the surface is plane, i.e. $r = \infty$, therefore,

$$\frac{|n_2 - n_1|}{r} = 0$$

Sign Convention:

(a) All rays of light are drawn travelling from left to right.

(b) All object distances (u) are considered positive when they are measured to the left of the pole and negative when they are measured to the right.

(c) All image distances (v) are positive when they are measured to the right of the pole and negative when to the left.

(d) Both focal lengths are positive for a converging system and negative for a diverging system.

(e) Object and image dimensions are positive when measured upwards from the optical axis and negative when measured downwards.

(f) All surfaces convex to the incident light from left to right are taken as having a positive radius, and all concave surfaces are taken as having a negative radius.

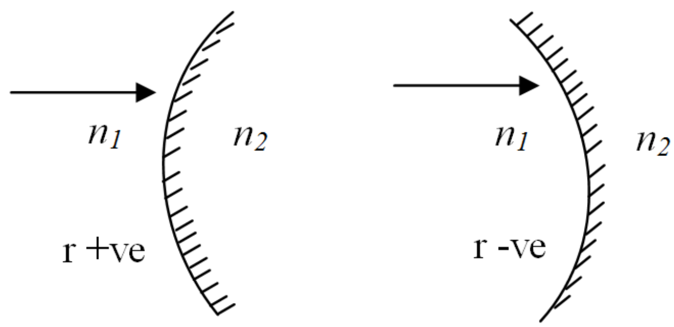


Fig. (43)

If the ray converges, the power $\frac{(n_2 - n_1)}{r}$ is positive and if the ray diverges, the power is negative.

Displacement of Lens When Object and Screen are Fixed:

Consider the following figure:

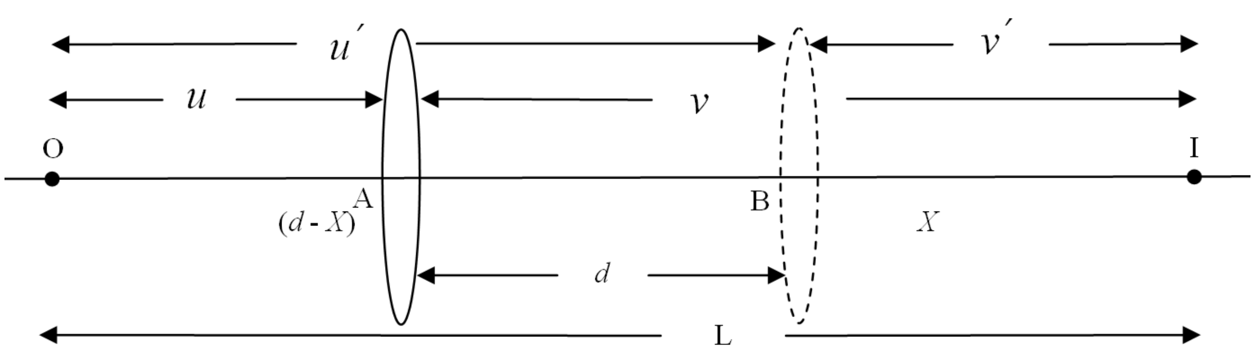


Fig. (44)

Since $v > u$, the image is larger than the object. If the points O and I are kept fixed another image at I can be obtained on the screen by moving the lens from position A to position B. This time the image is smaller than the object since BI < OB.

Since O and I are conjugate points, then

$$OB = IA \text{ and } IB = OA$$

$$\text{i.e. } \frac{1}{IB} + \frac{1}{OB} = \frac{1}{f} \text{ and } \frac{1}{IA} + \frac{1}{OA} = \frac{1}{f}$$

d = the displacement of the lens.

OI = L = constant

OA + BI = L – d

But OA = BI

$\therefore OA = \frac{(L-d)}{2}$

Also AI = AB + BI = AB + OA

$$\therefore AI = d + \frac{L-d}{2}$$

$$= \frac{L+d}{2}$$

But u = OA and v = AI at position A.

$$\therefore \frac{1}{(L+d)/2} + \frac{1}{(L-d)/2} = \frac{1}{f}$$

$$\frac{2}{(L+d)} + \frac{2}{(L-d)} = \frac{1}{f}$$

$$\text{Or } \left[f = \frac{L^2-d^2}{4L}\right]$$

This relation is useful for measuring the focal length when the lens is inaccessible where measurement of v and u cannot be made, i.g. when the lens is inside a telescope.

Remark:

If height of object = O, image 1 = I_1, and image 2 = I_2 then $\frac{O}{I_1} = \frac{u}{v}$ and $\frac{O}{I_2} = \frac{u'}{v'} = \frac{v}{u}$

$$\therefore \frac{O}{I_1} \times \frac{O}{I_2} = \frac{u}{v} \times \frac{v}{u} = 1$$

$$\therefore O^2 = I_1 I_2 \text{ or } O = \sqrt{I_1 I_2}$$

i.e. the height of the object is equal to the geometric mean of the two images.

Combined Focal Length of Two Thin Lenses in Contact:

Two combined thin lenses are used in telescopes and microscopes in order to diminish the coloring of the image due to dispersion when an object is viewed through a single lens.

Consider the following figure:

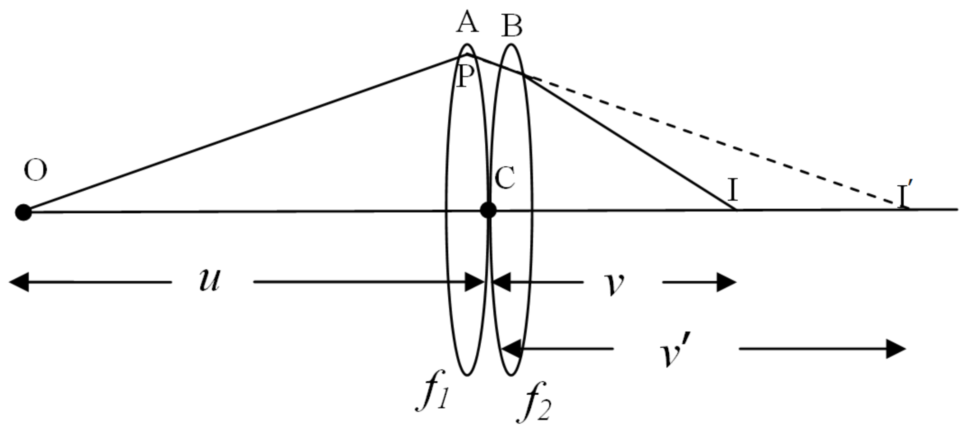


Fig. (45)

The thin lenses A and B of focal length f_1 and f_2 are in contact and their combined focal length is F.

Ray OP is refracted through the first lens A to intersect the optical axis at I′. thus I′ is the image of O in lens A.

OC = u and CI′ = v'

$$\therefore \frac{1}{u} + \frac{1}{v'} = \frac{1}{f_1} \ldots (1)$$

I' is a virtual object for lens B. The image is formed at I at distance CI = v from the lens B. Therefore, the object distance = $-v'$

$$\therefore \frac{1}{v} + \frac{1}{-v'} = \frac{1}{f_2} \ldots (2)$$

Adding equation (1) and (2) gives:

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2}$$

Since I is the image of O by refraction through both lenses, hence

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{F}$$

where F is the focal length of the combined lenses.

$$\therefore \frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$

This equation applies to any two thin lenses in contact. For example, if f_1 = +8 cm and f_2 = -12 cm, ∴ F = +24 cm.

The positive sign indicates that the combination acts like a converging lens.

Thin Lenses with Two Spherical Surfaces (Deviation of Lens Maker's Formula):

Consider the following equation of refraction at curved surfaces:

$$\frac{n_1}{u} + \frac{n_2}{v} = \frac{n_2 - n_1}{r}$$

For a thin lens at its thickness t is negligible, i.e. t → 0. Applying this equation, with the usual sign convention, on the first surface whose radius is r_1 as shown in figure (46)

$$\frac{n_1}{u} + \frac{n_2}{v'} = \frac{n_2 - n_1}{+r_1} \ldots (1)$$

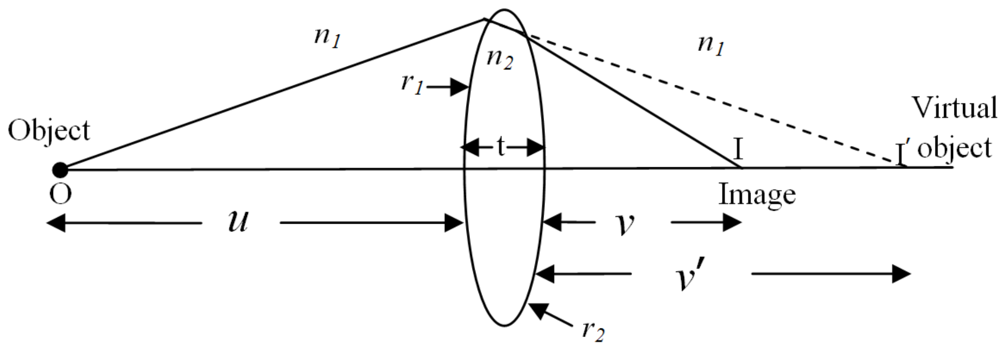


Fig. (46)

I' is the image of O formed by the first surface; it is the object for the second surface at distance v' on the right-hand side of the lens. Therefore, for the second surface whose radius is r_2 the equation becomes:

$$\frac{n_2}{-v'} + \frac{n_1}{v} = \frac{n_2 - n_1}{-(r_2)} \ldots (2)$$

Adding equation (1) and (2) gives:

$$\frac{n_1}{u} + \frac{n_1}{v} = \frac{n_2 - n_1}{r_1} - \frac{n_2 - n_1}{r_2}$$

$$\therefore \frac{n_1}{u} + \frac{n_1}{v} = (n_2 - n_1)\left[\frac{1}{r_1} - \frac{1}{r_2}\right] \ldots (3)$$

If $u = \infty$, $v = f_1$ (the first focal length)

$$\therefore \frac{n_1}{f_1} == (n_2 - n_1)\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

If $v = \infty$, $u = f_2$ (the second focal length)

$$\therefore \frac{n_1}{f_2} == (n_2 - n_1)\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$\therefore f_1 = f_2 = f$ since the media on the two sides of the lens are the same and of refractive index n_1.

$$\therefore \frac{n_1}{f} = (n_2 - n_1)\left[\frac{1}{r_1} - \frac{1}{r_2}\right] \ldots (4)$$

i.e. $$\frac{1}{f} = \left(\frac{n_2 - n_1}{n_1}\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

let n_2 the refractive index of the medium relative to air = n

$$\therefore \frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad \dots (5)$$

This is the Maker's formula; it applies to converging and diverging lenses.

Deviation Produced by a Lens:

The deviation produced by a lens, provided it is small, is independent of the angle of incidence of the rays and depends only on the vertical distance from the optical axis where the rays strike lens and its focal length.

Proof: First, consider a parallel ray incident at a height ***h*** from the optical axis on a convex lens of focal length ***f*** as shown in the figure below

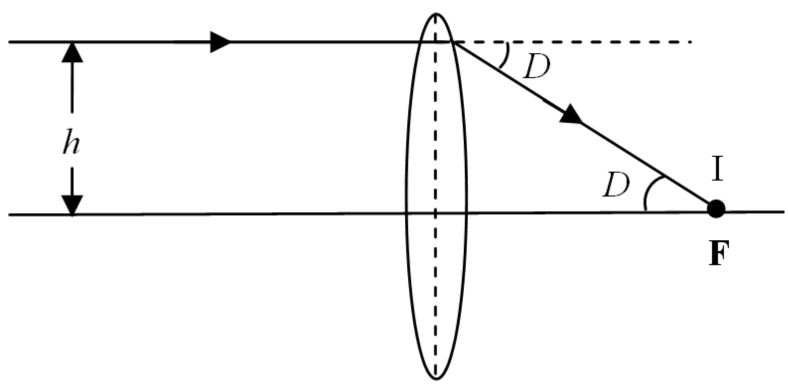


Fig. (47)

After reflection, the ray will converge to the principal focus **F** of the lens. The deviation produced is **D**. If the deviation angle **D** is small, then

$$D = \tan D = h/f \quad \dots (1)$$

Equation (I) shows that the deviation **D** depends on the height ***h*** and the focal length ***f*** but not upon the angle of incidence.

It can be proved that Eq. (1) is true even for oblique rays.

Second, consider the rays from a point object **O** as shown in Fig. (48).

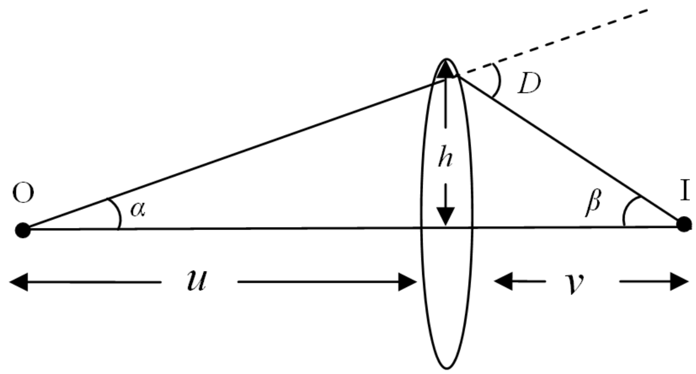


Fig. (48)

After refraction, these rays give rise to a point image **I**. the angle of deviation **D** is given by:

$$D = \alpha + \beta = \tan\alpha + \tan\beta$$

$$= \left(\frac{h}{u} + \frac{h}{v}\right) = h\left(\frac{1}{u} + \frac{1}{v}\right)$$

$$\therefore D = \frac{h}{f} \dots (2)$$

Equation (2) is the same as equation (1).

Equivalent Focal Length:

o there be two thin convex lenses of focal lengths f_1 and f_2 placed coaxially in air at a distance x from each other's shown in Fig. (49).

A ray of light **AB**, parallel to the optical axis is incident on the first lens $\mathbf{L_1}$ at a height of $\mathbf{h_1}$. It is deviated through an angle $\mathbf{D_1}$ and strikes the second lens $\mathbf{L_2}$ at height of $\mathbf{h_2}$ and suffers further deviation $\mathbf{D_2}$ in the same direction. Finally, it converges to a point **F**. Hence, point F is the focus of the combination.

It is required to find the focal length of a single lens which will form the image at the same point **F** as done by these two lenses together. Such lens is known as **equivalent lens** and its focal length as **equivalent focal length**.

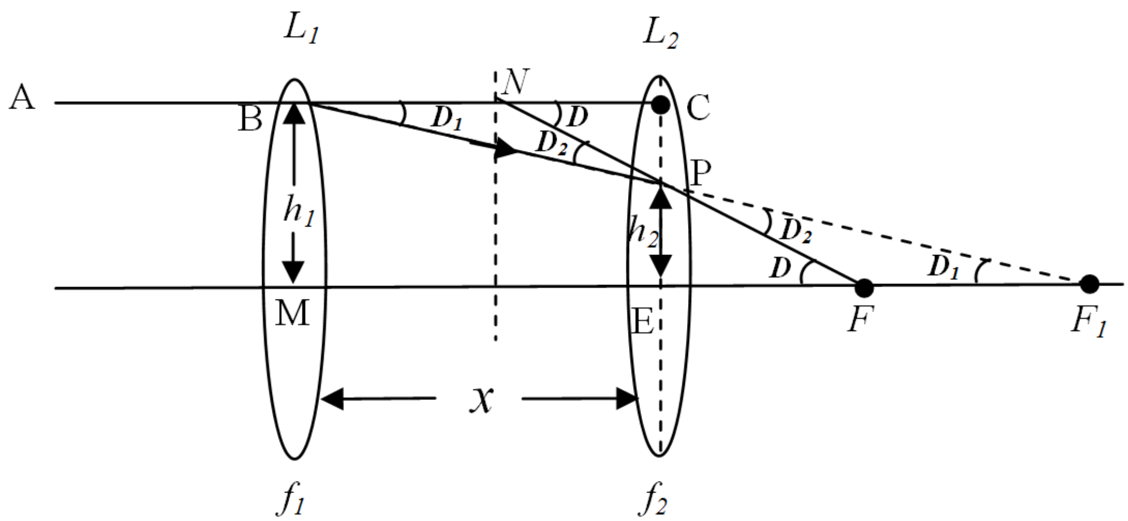


Fig. (49)

The total deviation D suffered by the ray is:

$$D = D1 + D2$$

where $D_1 = \frac{h_1}{f_1}$ and $D_2 = \frac{h_2}{f_2}$

$$\therefore D = \frac{h_1}{f_1} + \frac{h_2}{f_2}$$

where $h_2 = h_1 - CP$

if D_1 is small, then from ΔBCP,

$$D_1 = \frac{CP}{x}$$

$$CP = xD_1 = \frac{xh_1}{f_1}$$

$$\therefore\ h_2 = h_1 - \frac{xh_1}{f_1}$$

$$\therefore D = \frac{h_1}{f_1} + \left[\frac{h_1 - \frac{xh_1}{f_1}}{f_2}\right]$$

$$D = \frac{h_1}{f_1} + \frac{h_1}{f_2} - \frac{xh_1}{f_1 f_2}$$

If a single lens of focal length $f = LF$, be placed along the plane NL, it would focus the parallel rays (striking it at a height of $\mathbf{h_1}$) at point **F**. The deviation produced would be

$$D = \frac{h_1}{f}$$

$$\therefore \frac{h_1}{f} = \frac{h_1}{f_1} + \frac{h_1}{f_2} - \frac{xh_1}{f_1 f_2}$$

$$\therefore \frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{x}{f_1 f_2} \ldots (1)$$

The quantities $\frac{1}{f}, \frac{1}{f_1}$ and $\frac{1}{f_2}$, i.e. the reciprocal of the lens focal length, equal to the refractive power of the lens. If **f** is measured in meter then the power $P = 1 / f\,(m)$ has the unit of diopter, i.e.

$$P\ (diopters) = \frac{1}{f\ (meters)}$$

$$\text{and } P = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

If P_1 and P_2 are the powers of the two lenses, the power **P** of the equivalent lens is given by:

$$P = P_1 + P_2 - xP_1P_2$$

If the two lenses are in contact, i.e. $x = 0$, then

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

Or

$$\frac{1}{P} = \frac{1}{P_1} + \frac{1}{P_2}$$

If instead of two thin lenses, there are more of them in contact with each other, the equivalent focal length of the combination is given by:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \ \dots$$

It should be noted that equation (1) holds for convex and concave lenses provided proper sign is given to the focal length of each lens.