So, one has,

$$\frac{\partial \Psi}{\partial x} = \frac{i p_x}{\hbar} \underbrace{A\, e^{i(p_x x - E t)/\hbar}}_{\Psi(x,t)}$$

$$\Rightarrow \boxed{\hat{p}_x = -i\hbar \frac{\partial}{\partial x}} \;, \text{ Similarly } \boxed{\hat{E} = i\hbar \frac{\partial}{\partial t}}$$

$$\text{So, } \hat{T} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \quad \left\{ \hat{T} = \frac{\hat{p}_x^2}{2m} \right\}$$

Eigenvalue Equation: معادلة القيم الذاتية

For some function Ψ:

$$\hat{A}\Psi = \alpha \Psi$$

$$\text{or } \boxed{\hat{\ell}\Psi = \ell \Psi}$$

where α is a complex number. In this case, if Ψ is a member of class of physically meaningful functions, it is an eigenfunction of the operator $\hat{A}$. The number (α) is called the eigenvalue of $\hat{A}$ associated with the eigenfunction or state function Ψ.

Example 3: Show that the wave function $\psi = Nxe^{-\frac{x^2}{2}}$ is an eigen function of $H = -\frac{d^2}{dx^2} + x^2$

also, find out the normalization constant N and the eigen value.

$$H\psi = \left(-\frac{d^2}{dx^2} + x^2\right) Nx e^{-x^2/2}$$

$$= Nx^3 e^{-x^2/2} - N\frac{d}{dx}\left[e^{-x^2/2} - x e^{-x^2/2}\right]$$

$$= 3Nx e^{-x^2/2}$$

$$= 3\psi$$

Hence the eigenvalue is 3.

From normalization condition,

$$N^2 \int_{-\infty}^{+\infty} x^2 e^{-x^2} dx = 1$$

$$2N^2 \int_0^{\infty} x^2 e^{-x^2} dx = 1$$

$$x^2 = t \Rightarrow 2x\,dx = dt$$

$$x = t^{1/2} \Rightarrow dx = \frac{1}{2} t^{-1/2} dt$$

$$\Rightarrow 2N^2 \int_0^{\infty} t\, e^{-t} \cdot \frac{1}{2} t^{-\frac{1}{2}} dt = 1$$

$$2N^2 \cdot \frac{1}{2} \int_0^{\infty} t^{1/2} e^{-t} dt = 1$$

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx \quad \text{Gamma function}$$

$$\Gamma(\tfrac{3}{2}) = \int_0^\infty x^{\frac{1}{2}} e^{-x}\, dx \quad ; \quad \Gamma(p+1) = p\,\Gamma(p)$$

$$\Gamma(\tfrac{3}{2}) = \Gamma(\tfrac{1}{2}+1) = \tfrac{1}{2}\Gamma(\tfrac{1}{2})$$

$$= \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow N^2 \cdot \frac{\sqrt{\pi}}{2} = 1$$

$$N^2 = \frac{2}{\sqrt{\pi}} \Rightarrow N = \left(\frac{2}{\sqrt{\pi}}\right)^{1/2}$$

Hermitian Operators and their properties المؤثرات الهيرميتية

An operator $\hat{A}$ or $\hat{F}$, for example, ~~assoc~~ associated with a dynamical variable is said to be Hermitian if its average or expectation value in any eigenstate ψ is real.

The ~~avera~~ average value of an ~~operat~~ operator $\hat{A}$ for the normalized eigenfunction ψ is expressed as

$$\boxed{\langle A \rangle = \int \psi^* A \psi\, dx}$$

Or the Condition for an operator $\hat{A}$ to be Hermitian is_ تعريف مؤثر هرميتي

$$\boxed{\int_{\text{all space}} \psi_1^* \hat{A} \psi_2 \, dv = \int \psi_2 (\hat{A}\psi_1)^* dv}$$

مثال
<u>example</u>/ Show that the momentum operator $\hat{P}_x = -i\hbar \frac{\partial}{\partial x}$ is a Hermitian operator.

$$\int_{-\infty}^{+\infty} \psi_1^* \hat{A} \psi_2 \, dv = \int_{-\infty}^{+\infty} \psi_2 (\hat{A}\psi_1)^* dv$$

$$\int_{-\infty}^{+\infty} \psi_1^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi_2 \, dx = -i\hbar \int_{-\infty}^{+\infty} \underbrace{\psi_1^*}_{u} \underbrace{\frac{\partial}{\partial x} \psi_2 \, dx}_{dv}$$

$$\int u \, dv = uv - \int v \, du$$

let $\underline{u = \psi_1^*}$ $\quad \underbrace{du = \frac{\partial \psi_1^*}{\partial x} dx}$

$\underbrace{\frac{\partial}{\partial x} \psi_2 \, dx = dv}$ $\Rightarrow$ $\underline{v = \psi_2}$

$$\Rightarrow -i\hbar \int \psi_1^* \frac{\partial}{\partial x} \psi_2 \, dx = -i\hbar \left[\underbrace{\psi_1^* \psi_2 \Big|_{-\infty}^{+\infty}}_{0} - \int \psi_2 \frac{\partial \psi_1^*}{\partial x} dx \right]$$

$$= +i\hbar \int \psi_2 \frac{\partial}{\partial x} \psi_1^* \, dx$$

$$= \int \psi_2 \left(i\hbar \frac{\partial}{\partial x}\right) \psi_1^* \, dx$$