خواص المؤثر الهيرميتي

properties of Hermitian operators:

1. They have real eigenvalues.
2. Any set of mutually orthogonal eigenfunctions is linearly independent.

proof of property [1]: الخاصية الأولى

Let (f) be an eigenvalue of the Hermitian operator $\hat{A}$ in the state described by the normalized wavefunction ψ. The eigenvalue equation is:

$$\boxed{\hat{A}\psi = f\psi} \text{ ——— } (1)$$

its complex conjugate is:

$$\hat{A}^*\psi^* = f^*\psi^* \text{ ——— } (2)$$

multiplying eq. by ψ^*, and eq (2) by ψ, one has:

$$\psi^*\hat{A}\psi = \psi^* f \psi \text{ ——— } (3)$$

$$\psi\hat{A}^*\psi^* = \psi f^* \psi^* \text{ ——— } (4)$$

by integrating the eqs. (3), (4), respectively, one obtains:

$$\int \psi^* \hat{A} \psi \, dv = f \int \psi^* \psi \, dv$$

also,

$$\int \psi \hat{A}^* \psi^* \, dv = f^* \int \psi \psi^* \, dv$$

where $\psi_1 = \psi_2 = \psi$

But $\int \psi^* \psi \, dv = 1$ for a normalized wavefunction ψ,

$$f - f^* = 0 \Rightarrow \boxed{f^* = f}$$

which is possible only if (f) is a real number. Therefore, eigenvalue of Hermitian operator is real.

proof of property [2]: اثبات ثانية

Assume ψ_1, ψ_2 are two ~~[illegible]~~ eigenfunctions for the Hermitian operator $\hat{A}$ and their eigenvalues f_1, f_2 respectively, We apply the eigenvalue equation, that:

$$\hat{A}\psi_1 = f_1 \psi_1 \qquad (1)$$

$$\hat{A}\psi_2 = f_2 \psi_2 \qquad (2)$$

the complex conjugate of eq (1):

$$\hat{A}^* \psi_1^* = f_1^* \psi_1^*$$

from the property [1], the last equation becomes:

$$\hat{A}^* \psi_1^* = f_1 \psi_1^* \qquad (3)$$

Multiplying eq (3) by ψ_2 and eq. (2) by ψ_1^* that:

$$\psi_2 \hat{A}^* \psi_1^* = \psi_2 f_1 \psi_1^* \qquad (4)$$

$$\psi_1^* \hat{A} \psi_2 = \psi_1^* f_2 \psi_2 \qquad (5)$$

by integrating the eqs. (4), and (5) and subtracting one has:

$$\underbrace{\int \psi_2 \hat{A}^* \psi_1^* \, dv - \int \psi_1^* \hat{A} \psi_2 \, dv}_{\text{all space}} = (f_1 - f_2) \int \psi_2 \psi_1^* \, dv$$

according to definition of Hermitian operator $= 0$

$$\Rightarrow (f_1 - f_2) \int \psi_2 \psi_1^* \, dv = 0$$

$$f_1 \neq f_2 \quad \therefore \int \psi_2 \psi_1^* \, dv = 0$$

i.e., ψ_1, ψ_2 are linearly independent (Orthogonal wavefunctions) ; f_1, f_2 are different eigenvalues $(f_1 \neq f_2)$

حل

Example: Show that the operator $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ is a Hermitian operator. [illegible]

Sol.

تعريف،

$$\int_{-\infty}^{+\infty} \psi_1^* \hat{A} \psi_2 \, dv = \int_{-\infty}^{+\infty} \psi_2 (\hat{A} \psi_1)^* \, dx$$

the left-hand side:

$$-\frac{\hbar^2}{2m}\int \psi_1^* \frac{\partial^2}{\partial x^2} \psi_2 \, dx = ?!!$$

by integration by parts, one obtains:

$$-\frac{\hbar^2}{2m}\int \underbrace{\psi_1^*}_{u} \underbrace{\frac{\partial^2}{\partial x^2}\psi_2 \, dx}_{dv}$$

let $\psi_1^* = u \Rightarrow du = \frac{\partial \psi_1^*}{\partial x} dx \; ; \; dv = \frac{\partial^2}{\partial x^2}\psi_2 \, dx$

$\therefore \; v = \frac{\partial}{\partial x}\psi_2$

$\Rightarrow \int u \, dv = uv - \int v \, du$

$$-\frac{\hbar^2}{2m}\int \psi_1^* \frac{\partial^2}{\partial x^2}\psi_2\, dx = \underbrace{-\frac{\hbar^2}{2m}\psi_1^* \frac{\partial}{\partial x}\psi_2 \Big|_{-\infty}^{+\infty}}_{0} + \frac{\hbar^2}{2m}\int \frac{\partial}{\partial x}\psi_2 \frac{\partial \psi_1^*}{\partial x} dx$$

$$= \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \underbrace{\frac{\partial}{\partial x}\psi_2}_{dv}\, \underbrace{\frac{\partial \psi_1^*}{\partial x}}_{u}\, dx$$

Now,

$$u = \frac{\partial \psi_1^*}{\partial x}$$

$$du = \frac{\partial^2 \psi_1^*}{\partial x^2} dx$$

$$dv = \frac{\partial}{\partial x}\psi_2\, dx \Rightarrow v = \psi_2$$

$$\int u\, dv = uv - \int v\, du$$

$$\frac{\hbar^2}{2m}\int \frac{\partial}{\partial x}\psi_2 \frac{\partial \psi_1^*}{\partial x} dx = \underbrace{\frac{\partial \psi_1^*}{\partial x}\psi_2 \Big|_{-\infty}^{+\infty}}_{0} - \frac{\hbar^2}{2m}\int \psi_2 \frac{\partial^2}{\partial x^2}\psi_1^* dx$$

$$\therefore -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \psi_1^* \frac{\partial^2}{\partial x^2}\psi_2\, dx = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \psi_2 \left(\frac{\partial^2}{\partial x^2}\psi_1\right)^* dx$$

إذاً مؤثر الطاقة مؤثر هيرميتي

— Expectation value: القيم المتوقعة

The average of dynamical quantity (observable) is given by:

$$\langle A \rangle = \frac{\int \psi^* (A\psi)\, dv}{\int \psi^* \psi\, dv} \qquad (1)$$

if ψ is a normalized wavefunction, then

$$\int \psi^* \psi \, dv = 1$$ "normalization condition"

So, eq. (1) becomes:

$$\langle A \rangle = \int_{\text{all space}} \psi^* A \psi \, dv$$

Now, if the wavefunction ψ is ~~[illegible]~~ an eigen one ~~[illegible]~~ of the operator $\hat{a}$ with eigenvalue f, then:

$$\langle A \rangle = f \int \psi^* \psi \, dv \, ; \quad \int \psi^* \psi \, dv = 1$$

$$\therefore \boxed{\langle A \rangle = f}$$ i.e., the expectation value equals the eigen value.

* To evaluate the expectation value of P_x, one obtains:

$$\langle P_x \rangle = \int_{-\infty}^{+\infty} \psi^* \hat{P}_x \psi \, dx$$

or $$\langle P_x \rangle = -i\hbar \int_{-\infty}^{+\infty} \psi^* \frac{\partial}{\partial x} \psi \, dx$$

and so on, one can obtain the expectation value for any observable.

واجب بيتي

H.W Show that the wavefunction

$$\phi(x) = b\,e^{-ax} \quad ; a, b \text{ are constants}$$

is an eigen wavefunction of the operator $\frac{d^n}{dx^n}$.

Uncertainty Relations: علاقات اللاتحديد

The uncertainty in the position Δx can be expressed as,

$$\boxed{\Delta x = \left\{\langle x^2\rangle - \langle x\rangle^2\right\}^{1/2}}$$

Also, the uncertainty in the momentum coordinate Δp is expressed as,

$$\boxed{\Delta p_x = \left\{\langle p_x^2\rangle - \langle p_x\rangle^2\right\}^{1/2}}$$

for example,

$$\langle p_x^2\rangle = \int_{-\infty}^{\infty} \psi^*_{(x)}\, p_x^2\, \psi_{(x)}\, dx$$

$$p_x^2 = -\hbar^2 \frac{\partial^2}{\partial x^2} \Rightarrow \boxed{\langle p_x^2\rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi^*_{(x)} \frac{\partial^2}{\partial x^2} \psi_{(x)}\, dx}$$