$$\frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial x}\right) x\, e^{-y} \cdot e^{y} + \left(\frac{\partial z}{\partial x}\right)^2 \cdot e^{-2y} \cdot e^{2y}$$

$$\frac{\partial z}{\partial y} = x\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial x}\right)^2$$

PDE

Ex.(2) :- Eliminate h and K from the equation

$$(x-h)^2 + (y-K)^2 + z^2 = \alpha^2 \quad \text{---- (1)}$$

<u>Solution</u>

$$\left[2(x-h) + 2z\frac{\partial z}{\partial x} = 0\right] \div 2 \Rightarrow (x-h) + z\frac{\partial z}{\partial x} = 0$$

$$(x-h) = -z\frac{\partial z}{\partial x} \quad \text{---- (2)}$$

$$\left[2(y-K) + 2z\frac{\partial z}{\partial y} = 0\right] \div 2 \Rightarrow (y-K) + z\frac{\partial z}{\partial y} = 0$$

$$(y-K) = -z\frac{\partial z}{\partial y} \quad \text{---- (3)}$$

(2) and (3) in (1)

$$\left(-z\frac{\partial z}{\partial x}\right)^2 + \left(-z\frac{\partial z}{\partial y}\right)^2 + z^2 = \alpha^2$$

$$z^2\left(\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right)+z^2=\alpha^2$$

PDE

Ex. 3 : Eliminate a and b from the equation

$$2z=\frac{x^2}{a^2}+\frac{y^2}{b^2} \quad \text{---} \; (*)$$

<u>Solution</u>

$$\left[2\frac{\partial z}{\partial x}=2\frac{x}{a^2}\right]\div 2$$

$$\frac{\partial z}{\partial x}=\frac{x}{a^2} \quad \text{---}(1)$$

$$\left[2\frac{\partial z}{\partial y}=2\frac{y}{b^2}\right]\div 2$$

$$\frac{\partial z}{\partial y}=\frac{y}{b^2} \quad \text{---}(2)$$

- نعوض معادلة (1) و(2) في (*)

$$2z=\frac{x}{a^2}x+\frac{y}{b^2}y \Rightarrow 2z=\left(\frac{\partial z}{\partial x}\right)x+\left(\frac{\partial z}{\partial y}\right)y$$

PDE

Ex 4. **Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding partial diff. eqs**

1) $z = ax^3 + by^3$, (a and b) ...... ⊛

<u>Solution</u>

$$\left.\begin{aligned} \frac{\partial z}{\partial x} = 3ax^2 &\Rightarrow a = \frac{1}{3x^2}\cdot\frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} = 3by^2 &\Rightarrow b = \frac{1}{3y^2}\cdot\frac{\partial z}{\partial y} \end{aligned}\right\} \text{in } ⊛$$

$$z = ax^3 + by^3 \Rightarrow z = \left(\frac{1}{3\cancel{x^2}}\frac{\partial z}{\partial x}\right)x^{\cancel{3}\,1} + \left(\frac{1}{3\cancel{y^2}}\cdot\frac{\partial z}{\partial y}\right)y^{\cancel{3}\,1}$$

$$\Rightarrow \underbrace{z = \frac{x}{3}\cdot\frac{\partial z}{\partial x} + \frac{y}{3}\cdot\frac{\partial z}{\partial y}}_{\text{PDE}}$$

2) $4z = \left[ax + \frac{y}{a} + b\right]^2$, (a and b) ...... ⊛

<u>Solution</u>

$$\frac{\partial}{\partial x} \Rightarrow \left[4\frac{\partial z}{\partial x} = 2\left[ax + \frac{y}{a} + b\right]\cdot a\right] \quad \div 2a$$

$$\frac{2}{a}\cdot\frac{\partial z}{\partial x} = \left[ax + \frac{y}{a} + b\right] \quad \text{......} (1)$$

$$\frac{\partial}{\partial y} \Rightarrow \left[4\frac{\partial z}{\partial y} = 2\left[ax+\frac{y}{a}+b\right]\cdot\frac{1}{a}\right] * \frac{a}{2}$$

$$2a\frac{\partial z}{\partial y} = \left[ax+\frac{y}{a}+b\right] \text{ ---- } (2)$$

نعوض معادلة (1) و(2) في ⊛

$$4z = \left[ax+\frac{y}{a}+b\right]\left[ax+\frac{y}{a}+b\right]$$

$$4z = \left(\frac{2}{\cancel{a}}\cdot\frac{\partial z}{\partial x}\right)\left(2\cancel{a}\cdot\frac{\partial z}{\partial y}\right) \Rightarrow 4z = 4\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$$

$$\Rightarrow \underbrace{z = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)}_{PDE}$$

3) $z = ax^2 + bxy + cy^2 \quad (a, b, c)$ ---- ⊛

Solution

$$\frac{\partial}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = 2ax + by \text{ ---- } (1)$$

$$\frac{\partial^2}{\partial x^2} \Rightarrow \frac{\partial^2 z}{\partial x^2} = 2a \quad \Rightarrow \boxed{\frac{1}{2}\frac{\partial^2 z}{\partial x^2} = a}$$

$$\frac{\partial}{\partial y} \Rightarrow \frac{\partial z}{\partial y} = bx + 2cy \quad \text{---} (2)$$

$$\frac{\partial^2}{\partial y^2} \Rightarrow \frac{\partial^2 z}{\partial y^2} = 2C \quad \Rightarrow \boxed{\frac{1}{2}\frac{\partial^2 z}{\partial y^2} = C}$$

$$\frac{\partial^2}{\partial x \partial y} \Rightarrow \boxed{\frac{\partial^2 z}{\partial x \partial y} = b}$$

- نعوض قيمة a و b في معادلة (1)

$$\frac{\partial z}{\partial x} = 2ax + by \Rightarrow \underbrace{\frac{\partial z}{\partial x} = \left(\frac{\partial^2 z}{\partial x^2}\right)x + \left(\frac{\partial^2 z}{\partial x \partial y}\right)y}_{\text{PDE}}$$

- نعوض قيمة C و b في معادلة (2)

$$\frac{\partial z}{\partial y} = bx + 2cy \Rightarrow \underbrace{\frac{\partial z}{\partial y} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)x + \left(\frac{\partial^2 z}{\partial y^2}\right)y}_{\text{PDE}}$$

- نعوض قيمة a و b و c في معادلة ⊛

$$z = ax^2 + bxy + cy^2$$

$$\underbrace{z = \left(\frac{1}{2}\cdot\frac{\partial^2 z}{\partial x^2}\right)x^2 + \left(\frac{\partial^2 z}{\partial x \partial y}\right)xy + \left(\frac{1}{2}\cdot\frac{\partial^2 z}{\partial y^2}\right)y^2}_{\text{PDE}}$$

Methods for solving linear and non-linear partial differential equations of order one

طرق حل المعادلات التفاضلية الجزئية الخطية وغير الخطية من الرتبة الأولى

الطريقة رقم 1) طريقة لاكرانج :- وهو طريقة تحل المعادلة على شكل :-

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

طريقة الحل :-

1. نحدد P و Q و R من المعادلة الأصلية
2. نستعمل المعادلة المساعدة التالية :- $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
3. نأخذ أي معادلتين من المعادلة المساعدة و نحلها حسب طرق حل المرحلة الثانية لنستخرج قيم ثابتين أو أكثر مثل a, b.
4. نضع قيم a, b في المعادلة $\phi(a, b)=0$ ليكون هو الحل العام general solution.

Ex. 1 Solve $2\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = 2x$

<u>Solution:-</u>

$$P = 2 \ , Q = -3 \ , R = 2x$$

$$\frac{dx}{P} = \frac{dy}{q} = \frac{dz}{R} \Rightarrow \frac{dx}{2} = \frac{dy}{-3} = \frac{dz}{2x}$$

$$\frac{dx}{2} = \frac{dy}{-3} \Rightarrow \int -3\,dx = \int 2\,dy$$

$$\Rightarrow -3x = 2y + a \Rightarrow a = -3x - 2y$$

$$\frac{dx}{2} = \frac{dz}{2x} \Rightarrow \int x\,dx = \int dz$$

$$\Rightarrow \frac{x^2}{2} = z + b \Rightarrow b = \frac{x^2}{2} - z$$

$$\phi(a, b) = 0 \Rightarrow \phi\left(-3x - 2y, \frac{x^2}{2} - z\right) = 0$$

general solution

Ex.2 Solve $\left(\frac{y^2 z}{x}\right)p + xzq = y^2$

<u>Solution:-</u>

$$P = \frac{y^2 z}{x}, \quad Q = xz, \quad R = y^2$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{x}{y^2 z}dx = \frac{dy}{xz} = \frac{dz}{y^2}$$

$$\frac{x}{y^2 z}dx = \frac{dy}{xz} \Rightarrow \frac{x}{y^2}dx = \frac{dy}{x}$$

$$\Rightarrow \int x^2 dx = \int y^2 dy \Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + a$$

$$\Rightarrow a = \frac{x^3}{3} - \frac{y^3}{3}$$

$$\frac{x}{\cancel{y^2} z}dx = \frac{dz}{\cancel{y^2}} \Rightarrow \int x\,dx = \int dz$$

$$\Rightarrow \frac{x^2}{2} = z + b \Rightarrow b = \frac{x^2}{2} - z$$

$$\phi(a,b) = 0 \Rightarrow \underbrace{\phi\left(\frac{x^3}{3} - \frac{y^3}{3}, \frac{x^2}{2} - z\right) = 0}_{\text{general solution}}$$

Ex. 3 Solve $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+t\frac{\partial z}{\partial t}=xyt$

Solution

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{t}=\frac{dz}{xyt}$$

$$\int\frac{dx}{x}=\int\frac{dy}{y} \Rightarrow \ln x=\ln y+\ln a$$

$$\Rightarrow \ln x-\ln y=\ln a \Rightarrow \ln\frac{x}{y}=\ln a$$

$$\Rightarrow a=\frac{x}{y} \Rightarrow x=ay$$

$$\int\frac{dy}{y}=\int\frac{dt}{t} \Rightarrow \ln y-\ln t=\ln b$$

$$\Rightarrow \ln\frac{y}{t}=\ln b \Rightarrow b=\frac{y}{t} \Rightarrow t=\frac{y}{b}$$

$$\frac{dy}{\cancel{y}}=\frac{dz}{x\cancel{y}t} \Rightarrow xt\,dy=dz$$

$$\Rightarrow ay\left(\frac{y}{b}\right)dy=dz \Rightarrow \frac{a}{b}\int y^2dy-\int dz=\int 0$$

$$\Rightarrow \frac{a}{b}\frac{y^3}{3}-z=c$$

$$\Rightarrow \frac{x}{\cancel{y}}\cdot\frac{t}{\cancel{y}}\frac{y^{3}}{3}-z=c \Rightarrow c=\frac{xyt}{3}-z$$

$$\phi(a,b,c)=0 \Rightarrow \underbrace{\phi\left(\frac{x}{y}, \frac{y}{z}, \frac{xyz}{3}-z\right)=0}_{\text{general solution}}$$