Ex. 8 :- Find the particular solution

$$(D_x^3 - 7D_x D_y^2 - 6D_y^3)Z = x^2 . y$$

Solution :-

$$Z = \frac{1}{D_x^3 - 7D_x D_y^2 - 6D_y^3} x^2 . y$$

$$Z = \frac{1}{D_x^3\left(1 - \frac{7D_y^2}{D_x^2} - \frac{6D_y^3}{D_x^3}\right)} x^2 . y$$

$$Z = \frac{1}{D_x^3\left(1 - \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right)\right)} x^2 . y$$

$$Z = \frac{1}{D_x^3}\left(1 + \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right) + \cdots\right) x^2 . y$$

$$Z = \frac{1}{D_x^3}\left(x^2 y + 0 + 0\right)$$ because $D_y^n y^m = 0$ if $n > m$

$$Z = \frac{1}{D_x^3}(x^2 y) = \frac{1}{D_x^2}\left(\frac{x^3 y}{3}\right) = \frac{1}{D_x}\left(\frac{x^4 y}{12}\right) = \frac{x^5 y}{60}$$

$$\boxed{Z_p = \frac{x^5 y}{60}}$$ particular solution

Ex. 9 :- Find the <u>Particular solution</u>

$$\left(D_x^3 - a^2 D_x D_y^2\right) Z = x \quad , \text{where } a \in \mathbb{R}$$

Solution :-

$$Z = \frac{1}{D_x^3 - a^2 D_x D_y^2} x = \frac{1}{D_x^3\left(1 - \frac{a^2 D_y^2}{D_x^2}\right)} x$$

$$Z = \frac{1}{D_x^3}\left(1 + \left(\frac{a^2 D_y^2}{D_x^2}\right) + \cdots\right) x$$

$$Z = \frac{1}{D_x^3}\left(x + 0 + \cdots + 0\right)$$ because $D_y^n y^m = 0$ if $n >$

$$Z = \frac{1}{D_x^3} x = \frac{1}{D_x^2}\left(\frac{x^2}{2}\right) = \frac{1}{D_x}\left(\frac{x^3}{6}\right) = \frac{x^4}{24}$$

$$\boxed{Z_p = \frac{x^4}{24}}$$ Particular solution

$\boxed{\text{Case 4}}$ when $f(x,y) = e^{ax+by} \cdot V$ where V is a function of x and y.

$\boxed{\begin{array}{l} \sin(ax+by) \\ \cos(ax+by) \\ x^a \cdot y^b \end{array}}$

$$Z = \frac{1}{F(D_x, D_y)} e^{ax+by} \cdot V$$

$$Z = e^{ax+by} \frac{1}{F(D_x+a, D_y+b)} \cdot V$$

* نحله حسب الحالات السابقة

Ex. 10 :- Find the particular solution

$$D_x D_y Z = e^{2x+3y} \cdot x^2 y$$

solution :-

$$Z = \frac{1}{D_x D_y} e^{2x+3y} \cdot x^2 y$$

$$Z = e^{2x+3y} \frac{1}{(D_x+2)(D_y+3)} \cdot x^2 y$$

$$Z = e^{2x+3y} \cdot \frac{1}{3(D_x+2)(1+\frac{D_y}{3})} \cdot x^2 y$$

$$Z = \frac{e^{2x+3y}}{3} \cdot \frac{1}{(D_x+2)} \left(1 - \frac{D_y}{3} + \frac{D_y^2}{9} - \cdots\right) x^2 y$$

$$Z = \frac{e^{2x+3y}}{3} \cdot \frac{1}{(D_x+2)} \left(x^2 y - \frac{x^2}{3}\right)$$

$$Z = \frac{e^{2x+3y}}{6} \cdot \frac{1}{\left(1+\frac{D_x}{2}\right)} \left(x^2 y - \frac{x^2}{3}\right)$$

$$Z = \frac{e^{2x+3y}}{6} \left(1 - \frac{D_x}{2} + \frac{D_x^2}{4} - \frac{D_x^3}{8} + \cdots\right)\left(x^2 y - \frac{x^2}{3}\right)$$

$$Z = \frac{e^{2x+3y}}{6} \left(x^2 y - xy + \frac{y}{2} - \frac{x^2}{3} + \frac{x}{3} - \frac{1}{6}\right)$$

$$Z = e^{2x+3y} \left(\frac{x^2 y}{6} - \frac{xy}{6} + \frac{y}{12} - \frac{x^2}{18} + \frac{x}{18} - \frac{1}{36}\right)$$

Ex. 11 :- Find the particular solution

$$(D_x - D_y)^2 Z = e^{x+y} \sin(x+2y)$$

Solution :-

$$Z = \frac{1}{(D_x - D_y)^2} e^{x+y} \sin(x+2y)$$

$$Z = e^{x+y} \frac{1}{(D_x + \cancel{1} - D_y - \cancel{1})^2} \sin(x+2y)$$

$$Z = e^{x+y} \frac{1}{(D_x - D_y)^2} \sin(x+2y)$$

$$Z = e^{x+y} \frac{1}{D_x^2 - 2D_xD_y + D_y^2} \sin(x+2y)$$

$a = 1 \; , \; b = 2$

$$(-a^2, -ab, -b^2) = (-1, -2, -4)$$

$$F(-a^2, -ab, -b^2) = -1 + 4 - 4 = -1 \neq 0$$

$$Z_p = e^{x+y} \cdot \frac{1}{-1} \sin(x+2y)$$

$$Z_p = -e^{x+y} \cdot \sin(x+2y)$$

Case 5 when $f(x,y) = g(ax+by)$

ثانياً) $F(a,b) = 0$ ‏ ‏ ‏ ‏ أولاً) $F(a,b) \neq 0$

أولاً)) $F(a,b) \neq 0$

$$Z_p = \frac{1}{F(a,b)} \underbrace{\int \cdots \int}_{n\text{-times}} g(ax+by)\, \underbrace{d(ax+by) \ldots d(ax+by)}_{n\text{-times}}$$

* يعني تكامل $g(ax+by)$ من n من المرات حيث n تمثل رتبة المعادلة.

Ex. 12:- Find the <u>particular solution</u>

$$(D_x^2+2D_xD_y-8D_y^2)Z=\sqrt{2x+3y}$$

Solution:-

$$Z=\frac{1}{(D_x^2+2D_xD_y-8D_y^2)}\sqrt{2x+3y}$$

$a=2,\ b=3$, $g(2x+3y)=\sqrt{2x+3y}$

$F(a,b)=a^2+2ab-8b^2$

$F(2,3)=4+12-72=-56\neq 0$

$$Z=\frac{1}{F(a,b)}\iint g(ax+by)\,d(ax+by)\,d(ax+by)$$

$$Z=\frac{1}{-56}\iint(2x+3y)^{\frac{1}{2}}\,d(2x+3y)\,d(2x+3y)$$

$$Z=\frac{1}{-56}\int\frac{2}{3}(2x+3y)^{\frac{3}{2}}\,d(2x+3y)$$

$$Z=\frac{1}{-56}\left(\frac{4}{15}(2x+3y)^{\frac{5}{2}}\right)\Rightarrow Z_p=-\frac{1}{210}\sqrt{(2x+3y)^5}$$

ثانياً) $F(a,b)=0$

$F(a,b)=0$ then $F(D_x,D_y)=(bD_x-aD_y)^n$

$$Z_p=\frac{x^n}{n!}\cdot\frac{g(ax+by)}{b^n}$$

Ex. 13:- Find the particular solution

$$(D_x^2-6D_xD_y+9D_y^2)\,Z=3x+y$$

solution:-

$$Z=\frac{1}{(D_x^2-6D_xD_y+9D_y^2)}(3x+y)$$

$a=3$, $b=1$, $g(3x+y)=3x+y$

$F(a,b)=a^2-6ab+9b^2$

$F(3,1)=9-18+9=0$

$F(D_x,D_y)=(D_x^2-6D_xD_y+9D_y^2)=(D_x-3D_y)^2$

$n=2$

$$Z = \frac{x^2}{2!} \cdot \frac{3x+y}{1^2} \Rightarrow Z_p = \frac{x^2}{2}(3x+y)$$

Ex. 14:- Find the <u>particular solution</u>

$$(D_x^2 - D_y^2)Z = \sec^2(x+y)$$

Solution:-

$$Z = \frac{1}{D_x^2 - D_y^2}\sec^2(x+y)$$

$a = 1$, $b = 1$

$g(ax+by) = \sec^2(x+y)$

$$F(a,b) = F(1,1) = 1 - 1 = 0$$

$$F(D_x, D_y) = D_x^2 - D_y^2 = (D_x - D_y)(D_x + D_y)$$

$$Z = \frac{1}{(D_x - D_y)(D_x + D_y)}\sec^2(x+y)$$

$$u_1 = \frac{1}{(D_x + D_y)}\sec^2(x+y)$$

$a = 1$, $b = 1$

$g(ax+by) = \sec^2(x+y)$

$$F(a,b) = 1 + 1 = 2$$

$$u_1 = \frac{1}{2}\int \sec^2(x+y)\, d(x+y) = \frac{1}{2}\tan(x+y)$$

$$Z = \frac{1}{2(D_x - D_y)} \tan(x+y)$$

$a = 1 \; , b = 1$

$g(ax+by) = \tan(x+y)$

$F(a,b) = 1 - 1 = 0 \qquad , n = 1$

$$Z = \frac{1}{2} \cdot \frac{x^n}{n!} \cdot \frac{g(ax+by)}{b^n} \Rightarrow Z_p = \frac{1}{2} X . \tan(x+y)$$ الحل الخاص

4) $(D_x^2 + 3D_xD_y + 2D_y^2)Z = x+y.$

Sol:-

$m^2 + 3m + 2 = 0 \rightarrow (m+2)(m+1) = 0$

$m_1 = -2,\ m_2 = -1$

$Z = \phi_1(y + m_1x) + \phi_2(y + m_2x)$

$Z = \phi(y - 2x) + \phi_2(y - x).$

$$Z = \frac{1}{D_x^2 + 3D_xD_y + 2D_y^2}(x+y)$$

$$Z = \frac{1}{D_x^2\left(1 - \left(\frac{-3D_x}{D_x} - \frac{2D_y^2}{D_x^2}\right)\right)}(x+y)$$

$$Z = \frac{1}{D^2x}\left[x + y - \frac{3}{D_x}\right]$$

$$\frac{1}{D_x^2}[x + y - 3x] = \frac{1}{D_x}\left[\frac{x^2}{2} + yx - \frac{3x^2}{2}\right]$$

$$Z_2 = \frac{x^3}{6} + \frac{y}{2}x^2 - \frac{1}{2}x^3$$

$Z = Z_1 + Z_2$

$$= \phi_1(y - 2x) + \phi_2(y - x) + \frac{x^3}{6} + \frac{y}{2}x^2 - \frac{1}{2}x^3$$

ϕ_1, ϕ_2 arbitary function.

(5) $(D_x^2 - 5D_xD_y + 4D_y^2)Z = \sin(4x+y)$.

$m^2 - 5m + 4 = 0$

$\rightarrow (m-4)(m-1) = 0$

$m = 4, \; m = 1$

$Z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$

$Z = \phi(y+4x) + \phi_2(y+x)$

$Z = \dfrac{1}{D^2x - 5DxDy + 4D^2y} \sin(4x+y)$ $\qquad a=4, b=1$

$f(-a^2, -ab, -b^2) = -16 + 20 - 4 = 0$

$Z = \dfrac{1}{D^2x - 5DxDy + 4D^2y} \left(\dfrac{e^{4ix+yi} - e^{-4ix-iy}}{2i} \right)$ اويلر

$Z = \dfrac{1}{2i} \left(\dfrac{1}{D^2x - 5DxDy + 4D^2y} \cdot e^{4x+yi} \right) - \left(\dfrac{1}{D^2x - 5DxDy + 4D^2y} \cdot e^{-4ix+yi} \right)$

$u_1 = \dfrac{1}{D^2x - 5DxDy + 4D^2y} \cdot e^{4ix+yi}$ $\qquad a = 4i, b = i$

$f(a,b) = -(16i^2) + 20 - 4 = 16 + 20 - 4 = 32 \neq 0$

$u = \dfrac{1}{32} e^{4ix+yi}$

$u_2 =$