6. $(2D_x^2 - D_xD_y - 3D_y^2)Z = \frac{5e^x}{e^y}$

Sol:- $(2m^2 - m - 3) = 0$

$(2m - 3)(m + 1) = 0 \rightarrow m_1 = \frac{3}{2}, m_2 = -1$

$Z = \phi_1(y + \frac{3}{2}x) + \phi_2(y - x)$

$Z_2 = \frac{1}{2D_x^2 - D_xD_y - 3D_y^2} \, 5e^{x-y}$ $\quad a = 1, b = -1$

$f(a,b) = 2a^2 - ab - 3b^2$

$f(1,-1) = 2(1) + 1 - 3 = 0$

$f(D_x, D_y) = (2D_x - D_xD_y - 3D_y^2)$

$= \underbrace{(2D_x - 3D_y)}_{\neq 0}\underbrace{(D_x + D_y)}_{= 0}$

$Z = \frac{1}{\cancel{5}} \frac{x}{1} \cancel{5}e^{x-y} \rightarrow Z_1 = xe^{x-y}$

$Z = Z_1 + Z_2$

$Z = \phi(y + \frac{3}{2}x) + \phi_2(y - x) + xe^{x-y}$

ϕ_1, ϕ_2 arbitary function.

(7) $(D_x^2 - 3D_xD_y + 2D_y^2)z = e^{2x-y} + \cos(x+2y).$

$m^2 - 3m + 2 = 0 \rightarrow (m-2)(m-1)=0$

$m_1 = 2, \; m_2 = 1$

$Z_1 = \phi(y+2x) + \phi(y+x)$ g.s.

$$Z_2 = \frac{1}{D_x^2 - 3D_xD_y + 2D_y^2} e^{2x-y} + \frac{1}{D_x^2 - 3D_xD_y + 2D_y^2} \cos(x+2y)$$

$$u_1 = \frac{1}{D_x^2 - 3D_xD_y + 2D_y^2} e^{2x-y} = \quad f(a,b) = a^2 - 3ab + 2b^2$$

$$= 4 + 6 + 2$$

$$= 12 \neq 0$$

$$u_1 = \frac{1}{12} e^{2x-y}$$

$$u_2 = \frac{1}{D_x^2 - 3D_xD_y + 2D_y^2} \cos(x+2y). \qquad a=1, b=2$$

$$f(a,b) = a^2 - 3ab + 2b \rightarrow f(1,2) = 1 - 6 + 8 = 3 \neq 0$$

$$u_2 = \frac{1}{3} \iint \cos(x+2y) \, d(x+2y) \, d(x+2y)$$

$$u_2 = \frac{1}{3} \int \sin(x+2y) \, d(x+2y)$$

$$u_2 = \frac{-1}{3} \cos(x+2y)$$

$$Z = u_1 + u_2$$

$$Z_2 = \frac{1}{12} e^{2x-y} - \frac{1}{3} \cos(x+2y)$$

$$Z = Z_1 + Z_2 \rightarrow \phi(y+2x) + \phi_2(y+x) + \frac{1}{12} e^{2x-y} - \frac{1}{3} \cos(x+2y)$$

⑧ $(D_x^2 - D_x D_y) Z = \ln y.$

Sol:- $m^2 - m = 0 \rightarrow m(m-1)=0$

$m=0 \;,\; m-1=0 \rightarrow m=1$

$Z = \phi_1(y+m_1 x) + \phi_2(y+m_2 x)$

$Z = \phi_1(y) + \phi_2(y+x)$

P.I/ $Z = \dfrac{1}{D_x^2 - D_x D_y} \ln y \quad , a=0 \;, b=1$

$Z = \dfrac{1}{\underset{0=}{D_x}\,(\underset{\neq 0}{D_x - D_y})} \ln y$

$u_1 = \dfrac{1}{D_x - D_y} \ln y \qquad a=0 \;, b=1$

$f(a,b) = f(0,1) \rightarrow 0-1 = -1 \neq 0$

$u_1 = \dfrac{-1}{1} \int \ln y \; dy$

$du = \dfrac{1}{y} dy \rightarrow \int dv = \int dy \rightarrow v=y$

$u_1 = -1 \int \ln y \cdot u \int \frac{1}{y} \, dy$

$u_1 = -y \ln y + y$

$Z_2 = \dfrac{1}{D_x} (y - y\ln y) \qquad a=0 \;, b=1$

$f(0,1) = 0$

$Z_2 = \dfrac{x}{1} = y - y\ln y \qquad Z = x(y - y\ln y).$

(9) $(D_x + D_y) Z = \sec(x+y)$.

$m + 1 = 0 \rightarrow m = -1$

$Z = \phi(y - x)$

$Z = \dfrac{1}{D_x + D_y} \sec(x+y)$ $\quad a = 1,\ b = 1$

$f(a,b) = a + b \rightarrow f(1,1) = 1 + 1 = 2 \neq 0$ [illegible]

$Z_1 = \dfrac{1}{2} \int \sec(x+y)\, d(x+y)$

$Z_2 = \dfrac{1}{2} \ln|\sec(x+y) + \tan(x+y)|$

$Z = Z_1 + Z_2$

$= \phi(y - x) + \dfrac{1}{2} \ln|\sec(x+y) + \tan(x+y)|$

(6) $x(y^2-z^2)P + y(z^2-x^2)q = z(x^2-y^2)$

by Lagrange

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

$$\frac{x\,dx}{x^2(y^2-z^2)} = \frac{y\,dy}{y^2(z^2-x^2)} = \frac{z\,dz}{z^2(x^2-y^2)}$$

$$x^2(y^2-z^2) + y^2(z^2-x^2) + z^2(x^2-y^2) = 0$$

$$\cancel{x^2y^2} - \cancel{x^2z^2} + \cancel{y^2z^2} - \cancel{y^2x^2} + \cancel{z^2x^2} - \cancel{z^2y^2} = 0$$

$$x\,dx + y\,dy + z\,dz = 0 \rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a \quad \text{---} (1)$$

$$\frac{x^{-1}}{x^{-1}} \cdot \frac{dx}{x(y^2-z^2)} = \frac{y^{-1}}{y^{-1}} \cdot \frac{dy}{y(z^2-x^2)} = \frac{z^{-1}}{z^{-1}} \cdot \frac{dz}{z(x^2-y^2)}$$

$$\frac{x^{-1}\,dx}{y^2-z^2} = \frac{y^{-1}\,dy}{z^2-x^2} = \frac{z^{-1}\,dz}{x^2-y^2}$$

$$\cancel{y^2} - \cancel{z^2} + \cancel{z^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2} = 0$$

$$\rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz \rightarrow \ln x + \ln y + \ln z = 0$$

$$\boxed{xyz = b}$$

$$\phi(a,b) = 0 \rightarrow \phi\left(\frac{x^2}{2} + \frac{y^2}{2}, xyz\right) = 0$$

(11) $(y^2+z^2-x^2)p - 2xyq = -2xz$ by Lagrange.

$$\frac{dx}{y^2+z^2-x^2} = \frac{-dy}{2xy} = \frac{dz}{-2xz}$$

$$\frac{-dy}{2xy} = \frac{dz}{-2xz} \rightarrow \ln y = \ln z + \ln a$$

$$\frac{y}{z} = a \quad \text{---(1)} \quad \rightarrow y = az$$

$$\frac{dx}{y^2+z^2-x^2} = \frac{dz}{-2xz} \rightarrow -2xz\,dx = y^2dz + z^2dz - x^2dz$$

$$-2xz\,dx = a^2z^2dz + z^2dz - x^2dz$$

$$-x^2dz - 2xz\,dx = (a^2+1)z^2dz \quad] \div z^2$$

$$\left(\frac{2zx\,dx - x^2dz}{z^2}\right) = (a^2+1)dz$$

$$-d\left(\frac{x^2}{z}\right) = (a^2+1)dz$$

$$-\frac{x^2}{z} = (a^2+1)+b \rightarrow b = \frac{-x^2}{z} - \left(\frac{y^2}{z^2}+1\right)z$$

$$\phi(a,b)=0 \rightarrow \phi\left(\frac{y}{z}, \frac{-x^2}{z} - \left(\frac{y^2}{z^2}+1\right)z\right)=0$$

(12) $pq + 2y(x+1)q + x(x+2)q - 2(x+1) = 0$

$\div\ q(x+1)$

$$\frac{p}{x+1} + 2y + \frac{x(x+2)}{x+1} - \frac{2}{q} = 0$$

$$\frac{p}{x+1} + \frac{x(x+2)}{x+1} = \frac{2}{q} - 2y = a$$

$$\frac{p}{x+1} + \frac{x(x+2)}{x+1} = q$$

$$\rightarrow \frac{p}{x+1} = a - \frac{x(x+2)}{x+1}$$

$$p = a(x+1) - x(x+2)$$

$$\frac{2}{q} - 2y = a \rightarrow \frac{2}{q} = a - \frac{2}{y} \rightarrow q = \frac{2}{a+2y}$$

$$dz = (a(x-1) - x(x+2)\,dx + \frac{2}{a+2y}\,dy$$

$$z = \frac{ax^2}{2} + ax - \frac{x^3}{3} - x^2 + \ln|a+2y| + b$$

(13) $(x^2+2x)p+(x+1)qy=0$

Sol:- $(x^2+2x)p+(x+1)qy=0 \} \div (x+1)$

$$\frac{x^2+2x}{x+1}p+qy=0$$

$$\frac{x^2+2x}{x+1}p=-qy$$

$$\frac{x^2+2x}{x+1}p=a \rightarrow p=\frac{a(x+1)}{x^2+2x} \quad \text{—(1)}$$

$$-qy=a \rightarrow q=\frac{-a}{y} \quad \text{—(2)}$$

$$\int dz=\int p\,dx+\int q\,dy$$

$$\int dz=\int \frac{a(x+1)}{x^2+2x}dx+\frac{-a}{y}dy$$

$$z=\frac{a}{2}\ln|x^2+2x|-a\ln y+\ln a$$

(14)

$$m^2 + m - 2$$
$$m - 1 \,\big)\, m^3 - 3m + 2$$
$$\underline{m^3 - m^2}$$
$$m^2 - 3m$$
$$\underline{m^2 - m}$$
$$-2m + 2$$
$$\underline{-2m + 2}$$
$$0\,0$$

(14) $(D_x^3 - 3D_x D_y^2 + 2D_y^3)z = \dfrac{1}{\sqrt{3x-y}}$

$m^3 - 3m + 2 = 0$

$(m-1)(m^2 + m - 2)$

$\rightarrow (m-1)(m+2)(m-1) = 0$

$m_1 = m_2 = 1 \;,\; m_3 = -2$

$Z = \phi_1(y + m_1 x) + x\phi_2(y + m_2 x) + \phi_3(y + m_3 x)$

$Z = \phi(y + x) + x\phi(y + x) + \phi_3(y - 2x).$

$a = 3 \,,\; b = -1$

$f(3, -1) = 27 - 9 + 2 = 20 \neq 0$

$Z_2 = \dfrac{1}{20} \iiint (3x+y)^{\frac{-1}{2}} \, d(3x-y)\, d(3x-y)\, d(3x-y)$

$= \dfrac{2}{20} \iint (3x-y)^{\frac{2}{3}} \, d(3x-y)\, d(3x-y)$

$= \dfrac{2}{30} \int (3x-y)^{\frac{5}{2}} \, d(x-y)$

$= \dfrac{2}{75} (3x-y)^{\frac{5}{2}}$

$Z = Z_1 + Z_2$

$Z = \phi_1(y + x) + x\phi(y + x) + \phi_3(y - 2x) + \dfrac{2}{75}(3x-y)^{\frac{5}{2}}$

ϕ_1, ϕ_2, ϕ_3 arbitary function.

(15) $(D_x^3 + 2D_x^2 D_y - D_x D_y^2 - 2D_y^3)z = (y+2)e^x$

Sol:- $m^3 + 2m^2 - m - 2$

$(m-1)(m^2+3m+2) = 0$

$(m-1)(m+2)(m+1) = 0$

$m = 1,\ m = -2,\ m = -1$

$$\begin{array}{r|l} & m^2 + 3m + 2 \\ \hline m-1 & m^3 - 2m^2 - m - 2 \\ & \underline{m^3 - m^2} \\ & 3m^2 - m \\ & \underline{3m^2 - 3m} \\ & 2m - 2 \\ & \underline{2m - 2} \\ & 0 \quad 0 \end{array}$$

$z_1 = \phi(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x)$.

$z_1 = \phi(y+x) + \phi_2(y-2x) + \phi(y-x)$

$$z = \frac{1}{D_x^3 + 2D_x^2 D_y - D_x D_y^2 - 2D_y^3} \cdot (y+2)e^x$$

$$z = \frac{1}{(D_x+1+D_y)(D_x+1+2D_y)(D_x+1+D_y)} (y+2)$$

Let $u_1 = \frac{1}{D_x+1+D_y}(y+2)$

$u_1 = (1 - (D_x + D_y))(y+2)$

$u_1 = (y + 2 - 1) \rightarrow u_1 = y+1$

$$z_2 = e^x \frac{1}{(D_x+1+2D_y)(D_x+1+D_y)} (y+2)$$

$u_2 = \frac{1}{(1+(D_x+2D_y))}(y+2) \rightarrow u_2 = [1 - (D_x + 2D_y)](y+2)$

$u_2 = y - 1$

$z_2 = e^x \frac{1}{(1+D_x+D_y)}(y-1) \rightarrow z_2 = e^x(y+1-1) \rightarrow z_2 = ye^x$

$z = z_1 + z_2$

(16) $(4D_x^2 - 4D_xD_y + D_y^2)Z = (x+2y)^{\frac{3}{2}}$

Sol:- $4m^2 - 4m + 1 = 0$

$(2m-1)(2m-1) = 0$

$m_1 = m_2 = 2m - 1 = 0$

$m = \frac{1}{2}$

$Z = \phi_1(y + \frac{1}{2}x) + x\phi_2(y + \frac{1}{2}x)$

The P.I is Z_2

$g(ax+by) = (x+2y)^{3/2}$

$f(a,b) = 4a^2 - 4ab + b^2$

$= 4 - 8 + 4 = 0$

$f(D_x, D_y) = (4D_x^2 - 4D_xD_y + D_y^2)$

$= (2D_x - D_y)^2, \quad n = 2$

$Z_2 = \frac{x^2}{2!} \cdot \frac{(x+2y)^{3/2}}{2^2}$

$Z_2 = \frac{1}{8} x^2 (x+2y)^{\frac{3}{2}}$

$Z = Z_1 + Z_2$

$Z = \phi_1(y + \frac{1}{2}x) + x\phi(y + \frac{1}{2}x) + \frac{1}{8} x^2 (x+2y)^{3/2}$

ϕ_1, ϕ_2 arbitary function.

(17) $D_x D_y Z = e^{x-y} xy^2$

$Z_g = \phi(y)$

Sol: $m = 0 \rightarrow m_1 = 0$

P.I $\quad Z_p = \dfrac{1}{D_x D_y} e^{x-y} x^2 y$

$\rightarrow e^{x-y} \dfrac{1}{-(D_x+1)(1-D_y)} xy^2$

$\rightarrow e^{x-y} \dfrac{1}{-(D_x+1)} [1 + D_y + D_y^2 \dots] xy^2$

$\rightarrow -e^{x-y} \dfrac{1}{D_x+1} [xy^2 + 2xy^2 + 2x]$

$\rightarrow -e^{x-y} (1 - D_x + D_x^2 - \dots)(xy^2 + 2xy + 2x)$

$\rightarrow -e^{x-y} [xy^2 + 2xy + 2x - y^2 - 2y - 2]$

$Z = \phi(y) - e^{x-y} [xy^2 + 2xy + 2x - y^2 - 2y - 2]$

(18) $(D_x - D_y)Z = \tan(x+2y)$.

Sol: $m - 1 = 0 \rightarrow m = 1$

$Z = \phi(y+x)$

$$Z_2 = \frac{1}{D_x - D_y} \tan(x+2y)$$

$a = 1, \ b = 2 \qquad g(ax+by) = \tan(x+2y)$

$f(a,b) = a - b \rightarrow f(1,2) = 1 - 2 = -1 \neq 0$

$P.I = Z_2 = -\int \tan(x+2y) \cdot d(x+2y)$

$Z_2 = -\ln|\sec(x+2y)|$

$Z_2 = \ln(\sec(x+2y))^{-1}$

$Z = Z_1 + Z_2$

$Z = \phi(y+x) + \ln(\sec(x+2y))^{-1}$

(19) $2(D^3x - 9D^2xDy + 27DxD^2y - 27D^3y)Z = \tan^{-1}(3x+y)$

$m^3 - 9m^2 + 27D - 27 = 0$

$(m-3)(m^2+3m+9) - 9m^2 + 27m = 0$

$(m-3)(m^2+3m+9-9m) = 0 \rightarrow (m-3)^3 = 0$

$m_1 = m_2 = m_3 = 3$

$Z_g = \phi(y+3x) + \phi x(y+3x) + x^2\phi(y+3x).$

$$Zp = \frac{1}{D^3x - 9D^2xDy + 27DxD^2y} \cdot \frac{\tan^{-1}(3x+y)}{2}$$

$a = 3,\ b = 1 \rightarrow f(a,b) = a^2 - 9a^2b + 27ab^2 - 27b^3$

$f(3,1) = 27 - 81 + 81 - 27 = 0$

$f(DxDy) = (Dx - 3Dy)^3 = n = 3,\ b = 1$

$$Zp = \frac{x^3}{3!} \cdot \frac{\tan^{-1}(3x+y)}{1}$$

$$Zp = \frac{x^3\tan^{-1}(3x+y)}{6}$$

$Z = Z_1 + Z_2$

$$Z = \phi(y+3x) + \phi x(y+3x) + \phi x^2(y+3x) + \frac{x^3\tan^{-1}(3x+y)}{6}$$