Exercises page 33

1) $(Px)^2 = Z(Z - qy)$

Solution :-

$$X = \ln x \quad , \; xP = \frac{\partial Z}{\partial X} \text{ and } Y = \ln y \;, qy = \frac{\partial Z}{\partial Y}$$

$$\left(\frac{\partial Z}{\partial X}\right)^2 = Z\left(Z - \frac{\partial Z}{\partial Y}\right)$$

let $\frac{\partial Z}{\partial X} = t$ and $\frac{\partial Z}{\partial Y} = r$

$$t^2 = Z(Z - r)$$

the equation $t^2 = Z(Z-r)$ the form $f(t, r, Z) = 0$

taking $u = X + aY$

$$t = \frac{\partial Z}{\partial X} \cdot \frac{\partial u}{\partial u} = \frac{\partial Z}{\partial u} \cdot \frac{\partial u}{\partial X} = \frac{\partial Z}{\partial u}$$

$$r = \frac{\partial Z}{\partial Y} \cdot \frac{\partial u}{\partial u} = \frac{\partial Z}{\partial u} \cdot \frac{\partial u}{\partial Y} = a \frac{\partial Z}{\partial u}$$

$$\frac{\partial Z}{\partial u} = Z^2 - aZ\frac{\partial Z}{\partial u}$$

$$\frac{\partial Z}{\partial u} + aZ\frac{\partial Z}{\partial u} = Z^2$$

$$\frac{\partial Z}{\partial u}(1+aZ) = Z^2 \Rightarrow \int\left(Z^{-2} + \frac{a}{Z}\right)dZ = \int du$$

$$\Rightarrow \frac{-1}{Z} + a\ln Z = u + C$$

$$\Rightarrow -\frac{1}{Z} + a\ln Z = X + aY + C$$

$$\Rightarrow \boxed{\frac{-1}{Z} + a\ln Z = \ln x + a\ln y + C}$$

2) $\left[\,pq = z^2 y\sec x\,\right] \div z * z$

Solution

$$\frac{p}{z} * \frac{q}{z} = y\sec x$$

$$Z = \ln z \Rightarrow \quad \frac{p}{z} = \frac{\partial Z}{\partial x} \quad \text{and} \quad \frac{q}{z} = \frac{\partial Z}{\partial y}$$

$$\frac{\partial Z}{\partial x} * \frac{\partial Z}{\partial y} = y\sec x$$

let $\frac{\partial z}{\partial x} = t$ and $\frac{\partial z}{\partial y} = r$

$t \times r = y \sec x$

$\Rightarrow \frac{t}{\sec x} = \frac{y}{r} = a$

$\frac{t}{\sec x} = a \Rightarrow t = a \sec x$

$\frac{y}{r} = a \Rightarrow r = \frac{y}{a}$

$dz = t\,dx + r\,dy$

$\Rightarrow \int dz = a \int \sec x\,dx + \frac{1}{a} \int y\,dy$

$\Rightarrow z = a \ln|\sec x + \tan x| + \frac{1}{2a} y^2 + C$

$\Rightarrow \boxed{\ln z = a \ln|\sec x + \tan x| + \frac{1}{2a} y^2 + C}$

3) $p+q = z e^{x+y}$

Solution

$$\frac{p+q}{z} = e^x \cdot e^y \Rightarrow \frac{p}{z} + \frac{q}{z} = e^x \cdot e^y$$

$$Z = \ln z \Rightarrow \frac{p}{z} = \frac{\partial Z}{\partial x} \quad \text{and} \quad \frac{q}{z} = \frac{\partial Z}{\partial y}$$

$$\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} = e^x \cdot e^y$$

let $\frac{\partial Z}{\partial x} = t$ and $\frac{\partial Z}{\partial y} = r$

$$t + r = e^x \cdot e^y$$ ((لاكرانج))

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dZ}{e^x \cdot e^y}$$

$$dx = dy \Rightarrow x = y + a \Rightarrow a = x - y$$

$$\Rightarrow y = x - a$$

$$dx = \frac{dZ}{e^x \cdot e^y} \Rightarrow dx = \frac{dZ}{e^x \cdot e^{x-a}}$$

$$\Rightarrow dx = \frac{dZ}{e^{2x-a}} \Rightarrow \int dZ = \frac{1}{2}\int e^{2x-a} \, 2dx$$

$$\Rightarrow Z = \frac{1}{2}e^{2x-a} + b \Rightarrow b = Z - \frac{1}{2}e^{2x-a}$$

$$\phi(a, b) = 0$$

$$\Rightarrow \phi\left(x-y, Z - \frac{1}{2}e^{2x-a}\right) = 0$$

$$\Rightarrow \boxed{\phi\left(x-y, \ln z - \frac{1}{2}e^{2x-a}\right) = 0}$$

4) $p^2 + zq = z^2(x-y)$

<u>Solution</u>

$$\left(\frac{p}{z}\right)^2 + \frac{q}{z} = x-y$$

$$Z = \ln z \Rightarrow \quad \frac{p}{z} = \frac{\partial Z}{\partial x} \quad \text{and} \quad \frac{q}{z} = \frac{\partial Z}{\partial y}$$

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \frac{\partial Z}{\partial y} = x-y$$

let $\frac{\partial Z}{\partial x} = t$ and $\frac{\partial Z}{\partial y} = r$

$$t^2 + r = x - y \Rightarrow r + y = x - t^2 = a$$

$$x - t^2 = a \Rightarrow t^2 = x - a \Rightarrow t = \mp\sqrt{x-a}$$

$$r + y = a \Rightarrow r = a - y$$

$$dZ = t\,dx + r\,dy$$

$$\Rightarrow \int dZ = \mp \int (x-a)^{\frac{1}{2}} dx + \int (a-y)\,dy$$

$$\Rightarrow Z = \mp \frac{2}{3} \sqrt{(x-a)} + ay - \frac{y^2}{2} + C$$

$$\Rightarrow \boxed{\ln z = \mp \frac{2}{3} \sqrt{(x-a)} + ay - \frac{y^2}{2} + C}$$

5) $p^2 + q^2 = z^2$

<u>Solution</u> :-

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = 1$$

$$Z = \ln z \Rightarrow \quad \frac{p}{z} = \frac{\partial Z}{\partial x} \quad \text{and} \quad \frac{q}{z} = \frac{\partial Z}{\partial y}$$

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = 1$$

let $t = \dfrac{\partial Z}{\partial x}$ and $r = \dfrac{\partial Z}{\partial y}$

$$t^2 + r^2 = 1$$

from $f(t, r) = 0$

let $t = a$ and $r = b$

$$a^2 + b^2 = 1 \Rightarrow a^2 = 1 - b^2 \Rightarrow a = \mp\sqrt{1-b^2}$$

from $dZ = a\,dx + b\,dy$

$$\Rightarrow \int dZ = \int a\,dx \mp \int \sqrt{1-b^2}\,dy$$

$$\Rightarrow Z = ax \mp \sqrt{1-b^2}\,y + C$$

$$\Rightarrow \boxed{\ln z = ax \mp \sqrt{1-b^2}\,y + C}$$

6) $xP + 4q = \cos y$

Solution:-

$X = \ln x$, $xP = \dfrac{\partial z}{\partial X}$

$\dfrac{\partial z}{\partial X} + 4q = \cos y$, let $t = \dfrac{\partial z}{\partial X}$

$$t + 4q = \cos y$$

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$$\frac{dX}{1} = \frac{dy}{4} = \frac{dZ}{\cos y}$$

$$\int dX = \int \frac{1}{4} dy \Rightarrow X = \frac{y}{4} + a \Rightarrow a = X - \frac{y}{4}$$

$$\frac{dy}{4} = \frac{dZ}{\cos y} \Rightarrow \int \cos y \, dy = \int 4 \, dZ$$

$$\Rightarrow \sin y = 4Z + b \Rightarrow b = \sin y - 4Z$$

$$\phi(a, b) = 0 \Rightarrow \phi\left(X - \frac{y}{4}, \sin y - 4Z\right) = 0$$

$$\Rightarrow \boxed{\phi\left(\ln x - \frac{y}{4}, \sin y - 4Z\right) = 0}$$