ملاحظة :- إذا كان مجموع مقامات المعادلة المساعدة يساوي صفر ، فإن مجموع بسوط المعادلة المساعدة أيضاً يساوي صفر

- في السؤال السابق

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{xyt}$$

$$\frac{yt}{yt} \cdot \frac{dx}{x} = \frac{xt}{xt} \cdot \frac{dy}{y} = \frac{xy}{xy} \cdot \frac{dt}{t} = \frac{-3}{-3} \cdot \frac{dz}{xyt}$$

$$xyt + xyt + xyt - 3xyt = 3xyt - 3xyt = 0$$

إذن مجموع البسوط أيضاً = صفر

$$yt\,dx + xt\,dy + xy\,dt - 3\,dz = 0$$

$$\int d(xyt) - \int 3dz = \int 0$$

$$xyt - 3z = a$$

- وبهذا الشكل نستطيع استخراج أحد الثوابت.

Ex. 4 Solve $(y-z)p+(z-x)q=x-y$

Solution

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}$$

$$y-z+z-x+x-y=0$$

then $\int dx+\int dy+\int dz=\int 0$

$$\Rightarrow x+y+z=a$$

- لاستخراج قيمة الـ b نضرب المعادلة المساعدة بـ x, y, z
بحيث يبقى المقام = صفر.

$$\frac{x}{x}\frac{dx}{y-z}=\frac{y}{y}\frac{dy}{z-x}=\frac{z}{z}\frac{dz}{x-y}$$

$$xy-xz+yz-xy+xz-yz=0$$

then $\int x\,dx+\int y\,dy+\int z\,dz=\int 0$

$$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=b$$

$$\phi(a,b)=0 \Rightarrow \phi\left(x+y+z, \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}\right)=0$$

Solve the following partial differential equation

1) $\tan x \, p + \tan y \, q = \tan z$

Solution

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} \Rightarrow \int \frac{\cos x}{\sin x}\,dx - \int \frac{\cos y}{\sin y}\,dy = \int 0$$

$$\Rightarrow \ln|\sin x| - \ln|\sin y| = \ln a \Rightarrow \ln \frac{\sin x}{\sin y} = \ln a$$

$$\Rightarrow a = \frac{\sin x}{\sin y}$$

$$\frac{dy}{\tan y} = \frac{dz}{\tan z} \Rightarrow \int \frac{\cos y}{\sin y}\,dy - \int \frac{\cos z}{\sin z}\,dz = \int 0$$

$$\Rightarrow \ln|\sin y| - \ln|\sin z| = \ln b \Rightarrow \ln \frac{\sin y}{\sin z} = \ln b$$

$$\Rightarrow b = \frac{\sin y}{\sin z}$$

$$\phi(a, b) = 0 \Rightarrow \underbrace{\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0}_{\text{general solution}}$$

2) $y^2 p - xyq = x(z-2y)$

<u>Solution</u>

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dx}{y^{\cancel{2}1}} = \frac{dy}{-x\cancel{y}} \Rightarrow -\int x\,dx = \int y\,dy$$

$$\Rightarrow -\frac{x^2}{2} - \frac{y^2}{2} = a$$

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)} \Rightarrow \frac{dy}{-y} = \frac{dz}{z-2y}$$

$$\Rightarrow (z-2y)\,dy = -y\,dz \Rightarrow (z-2y)\,dy + y\,dz = 0$$

$$\Rightarrow y\frac{dz}{dy} + z - 2y = 0 \Rightarrow \frac{dz}{dy} + \frac{z}{y} = 2$$

معادلة خطية

$f(y) = \frac{1}{y}$, $g(y) = 2$

$$I = e^{f(y)\,dy} = e^{\frac{1}{y}\,dy} = e^{\ln y} \Rightarrow I = y$$

$$Iz = \int I\,g(y)\,dy + b$$

$$yz = \int 2y\,dy + b \Rightarrow yz = y^2 + b$$

$$\Rightarrow b = yz - y^2$$

$$\phi(a,b) = 0 \Rightarrow \underbrace{\phi\left(-\frac{x^2}{2} - \frac{y^2}{2}, yz - y^2\right) = 0}_{\text{general solution}}$$

3) $xp + yq = z$

<u>Solution</u>

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y} \Rightarrow \ln x - \ln y = \ln a$$

$$\Rightarrow \ln \frac{x}{y} = \ln a \Rightarrow a = \frac{x}{y}$$

$$\frac{dx}{x} = \frac{dz}{z} \Rightarrow \ln x - \ln z = \ln b$$

$$\Rightarrow \ln \frac{x}{z} = \ln b \Rightarrow b = \frac{x}{z}$$

$$\phi(a,b) = 0 \Rightarrow \phi\left(\frac{x}{y}, \frac{x}{z}\right) = 0 \Big\} \text{general solution}$$

4) $(-a+x)p+(-b+y)q=(-c+z)$

solution

$$\frac{dx}{-a+x}=\frac{dy}{-b+y}=\frac{dz}{-c+z}$$

$$\int\frac{dx}{-a+x}=\int\frac{dy}{-b+y} \Rightarrow \ln(-a+x)-\ln(-b+y)=\ln h$$

$$\Rightarrow \ln\left(\frac{-a+x}{-b+y}\right)=\ln h \Rightarrow h=\frac{-a+x}{-b+y}$$

$$\int\frac{dx}{-a+x}=\int\frac{dz}{-c+z} \Rightarrow \ln(-a+x)-\ln(-c+z)=\ln K$$

$$\Rightarrow \ln\left(\frac{-a+x}{-c+z}\right)=\ln K \Rightarrow K=\frac{-a+x}{-c+z}$$

$$\phi(h,K)=0 \Rightarrow \underbrace{\phi\left(\frac{-a+x}{-b+y},\frac{-a+x}{-c+z}\right)=0}_{\text{general solution}}$$

5) $x^2p+y^2q=-z^2$

solution

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{-z^2}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \int x^{-2}dx = \int y^{-2}dy$$

$$\Rightarrow -\frac{1}{x} + \frac{1}{y} = a$$

$$\frac{dx}{x^2} = \frac{dz}{-z^2} \Rightarrow \int x^{-2}dx = -\int z^{-2}dz$$

$$-\frac{1}{x} - \frac{1}{z} = b$$

$$\phi(a,b) = 0 \Rightarrow \underbrace{\phi\left(-\frac{1}{x}+\frac{1}{y}, -\frac{1}{x}-\frac{1}{z}\right) = 0}_{\text{general solution}}$$

6) $yzp + zxq = xy$

<u>solution</u>

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

$$\frac{dx}{yz} = \frac{dz}{xy} \Rightarrow \int x\,dx = \int z\,dz$$

$$\Rightarrow \frac{x^2}{2} - \frac{z^2}{2} = a$$

$$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow \int y\,dy = \int z\,dz$$

$$\frac{y^2}{2} - \frac{z^2}{2} = b$$

$$\phi(a,b) = 0 \Rightarrow \underbrace{\phi\left(\frac{x^2}{2} - \frac{z^2}{2}, \frac{y^2}{2} - \frac{z^2}{2}\right) = 0}_{\text{general solution}}$$

7) $y^2 p + x^2 q = x^2 y^2 z^2$

Solution

$$\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2}$$

$$\frac{dx}{y^2} = \frac{dy}{x^2} \Rightarrow \int x^2\,dx = \int y^2\,dy \Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = a$$

$$\frac{dx}{\cancel{y^2}} = \frac{dz}{x^2 \cancel{y^2} z^2} \Rightarrow \int x^2\,dx = \int z^{-2}\,dz \Rightarrow \frac{x^3}{3} + \frac{1}{z} = b$$

$$\phi(a,b) = 0 \Rightarrow \underbrace{\phi\left(\frac{x^3}{3} - \frac{y^3}{3}, \frac{x^3}{3} + \frac{1}{z}\right) = 0}_{\text{general solution}}$$

8) $p - q = \frac{z}{x+y}$

Solution

$$dx = -dy = \frac{x+y}{z}\,dz$$

$$\int dx = -\int dy \Rightarrow x + y = a \Rightarrow \boxed{y = a - x}$$

$$dx = \frac{x+y}{z}\,dx \Rightarrow dx = \frac{x+a-x}{z}\,dz$$

$$\Rightarrow \int dx = \int \frac{a}{z}\,dz \Rightarrow x = a\ln z + b$$

$$\Rightarrow b = x - (x+y)\ln z$$

$$\phi(a,b) = 0 \Rightarrow \underbrace{\phi(x+y,\ x-(x+y)\ln z) = 0}_{\text{general solution}}$$

الطريقة رقم 2 :- إذا كانت المعادلة على شكل $f(p,q)=0$
حيث تكون غير خطية ومن الرتبة الاولى
تكون طريقة الحل :-

1. نفرض ان $p=a$ و $q=b$ ونعوض في المعادلة الأصلية ثم نستخرج اما a بدلالة b أو العكس.

2. من هذه المعادلة $dz=adx+bdy$ بالتكامل نحصل على
$z=ax+by+c$

3. نعوض قيمة a أو b المستخرجة من الخطوة الاولى في المعادلة $z=ax+by+c$ ليكون هو الحل التام ((complete solution))

Ex.1 :- Solve $p^2+p=q^2$

Solution

Let $p=a,\ q=b \Rightarrow a^2+a=b^2$

$\Rightarrow b=\mp\sqrt{a^2+a}$

from $z=ax+by+c$

$$\Rightarrow \underbrace{z=ax+\left(\mp\sqrt{a^2+a}\right)y+c}_{\text{complete solution}}$$

Ex. 2 :- solve $pq = k$, where k is a constant

Solution

let $p = a$, $q = b \Rightarrow ab = k \Rightarrow a = \frac{k}{b}$

from $z = ax + by + c \Rightarrow z = \frac{k}{b}x + by + c$

complete solution

Ex. 3 :- Solve $p - 3q = q^3$

Solution

let $p = a$, $q = b \Rightarrow a - 3b = b^3$

$\Rightarrow a = b^3 + 3b$

from $z = ax + by + c \Rightarrow z = (b^3 + 3b)x + by + c$

complete solution

الطريقة رقم 3 :- إذا كان شكل المعادلة $z = xp + yq + f(p,q)$
يكون الحل التام بتعويض $q=b$ and $p=a$ فقط
$$z = ax + by + f(a,b).$$

Ex.1 :- Solve $z = px + qy + pq$

Solution

let $p = a \; , q = b$

$\Rightarrow z = ax + by + ab \rightarrow$ complete solution

Ex.2 :- Solve $xp + yq = z - 5p + pq$

Solution

$$z = xp + yq + 5p - pq$$

Let $p = a \; , q = b$

$z = xa + yb + 5a - ab \rightarrow$ complete solution

Ex.3 :- solve $px + qy = z - p^3 - q^3$

Solution

$$z = px + qy + p^3 + q^3$$

Let $p = a \; , q = b$

$z = ax + by + a^3 + b^3 \rightarrow$ complete solution