Homogeneous linear PDEs with constant coefficients and higher order

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$$A_0 \frac{\partial^n Z}{\partial x^n} + A_1 \frac{\partial^n Z}{\partial x^{n-1} \cdot \partial y} + \cdots + A_n \frac{\partial^n Z}{\partial y^n} = f(x,y)$$

الشكل العام

where $A_0, A_1, \ldots, A_n$ are constant coefficients

For example

1) $3Z_{xx} + 5Z_{xy} + Z_{yy} = 0$ homo. of order 2

2) $2Z_{xxx} - 3Z_{xxy} + 5Z_{xyy} - 8Z_{yyy} = x+y$ homo. of order 3

((نعرف من المرحلة الثانية بان))

$$\frac{\partial}{\partial x^n} = D_x^n \quad \text{and} \quad \frac{\partial}{\partial y^n} = D_y^n$$

$$A_0 D_x^n Z + A_1 D_x^{n-1} D_y Z + \cdots + A_n D_y^n Z = f(x,y)$$

$$\Rightarrow \left(A_0 D_x^n + A_1 D_x^{n-1} D_y + \dots + A_n D_y^n\right) Z = f(x,y)$$

$$\Rightarrow F(D_x, D_y) Z = f(x,y)$$

general solution ↙ ↘ Particular solution

1) $f(x,y) = 0$ 2) $f(x,y) \neq 0$

- المعادلة المميزة (لاستخراج الحل العام general solution)

$$F(D_x, D_y) Z = 0 \quad \text{where } \underline{f(x,y) = 0}$$

فقط نعوض بدل $D_x = m$ و $D_y = 1$.

$$F(m, 1) = 0$$

ثم نستخرج قيم m (جذور المعادلة)

عقدية (تخيلية) | حقيقية متشابها (مكرره) | حقيقية مختلفة

1) when the roots are **distinct**.

If $m_1, m_2, \dots, m_n$

then $Z = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \dots + \phi_3(y+m_nx)$

2) when the roots are **repeated**.

If $m_1 = m_2 = \dots = m_K$

then $Z = \phi_1(y+m_1x) + x\phi_2(y+m_1x) + \dots + x^{K-1}\phi_n(y+m_1x)$

3) when the roots are **complex**.

If $m_1, m_2, \dots, m_n$

then $Z = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \dots + \phi_3(y+m_nx)$

Ex. 1 :- Find the general solution of

$$(D_x^3 + 2D_x^2 D_y - 5D_x D_y^2 - 6D_y^3)Z = 0$$

Solution :-

$$m^3 + 2m^2 - 5m - 6 = 0$$

$m = -1$ يحقق المعادلة

$$m + 1 = 0$$

$$\begin{array}{r|l} & m^2 + m - 6 \\ \hline m+1 & m^3 + 2m^2 - 5m - 6 \\ & \underline{m^3 + m^2} \quad - \\ & m^2 - 5m - 6 \\ & \underline{m^2 + m} \quad - \\ & -6m - 6 \\ & \underline{-6m - 6} \quad - \\ & 0 \end{array}$$

$$(m+1)(m^2 + m - 6) = 0$$

$$(m+1)(m+3)(m-2) = 0$$

$$m_1 = -1 \quad m_2 = -3 \quad m_3 = 2$$

$$Z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x)$$

$$\boxed{Z = \phi_1(y - x) + \phi_2(y - 3x) + \phi_3(y + 2x)}$$

general solution

Ex. 2:- Find the general solution of

$$\left(D_x^3 - D_x^2 D_y - 8 D_x D_y^2 + 12 D_y^3\right) Z = 0$$

solution:-

$$m^3 - m^2 - 8m + 12 = 0$$

$m = 2$ يتحقق المعادلة

$$m - 2 = 0$$

$$\begin{array}{r|l} & m^2 + m - 6 \\ \hline m-2 & m^3 - m^2 - 8m + 12 \\ & m^3 - 2m^2 \quad - \\ \hline & m^2 - 8m + 12 \\ & m^2 - 2m \quad - \\ \hline & -6m + 12 \\ & -6m + 12 \quad - \\ \hline & 0 \end{array}$$

$$(m-2)(m^2 + m - 6) = 0$$

$$(m-2)(m-2)(m+3) = 0$$

$$m_1 = m_2 = 2 \,, \quad m_3 = -3$$

$$Z = \phi_1(y + m_1 x) + x\phi_2(y + m_2 x) + \phi_3(y + m_3 x)$$

$$Z = \phi_1(y + 2x) + x\phi_2(y + 2x) + \phi_3(y - 3x)$$

Ex. 3:- Find the general solution of

$$(m-1)^2 (m+2)^3 (m-3)(m+4) = 0$$

Solution:-

$$m_1 = m_2 = 1 \,, \quad m_3 = m_4 = m_5 = -2 \,, \quad m_6 = 3 \,, \quad m_7 = -4$$

$$Z = \phi_1(y + m_1 x) + x\phi_2(y + m_2 x) + \phi_3(y + m_3 x) + x\phi_4(y + m_4 x) +$$

$$x^2\phi_5(y+m_5x)+\phi_6(y+m_6x)+\phi_7(y+m_7x)$$

$$Z=\phi_1(y+x)+x\phi_2(y+x)+\phi_3(y-2x)+x\phi_4(y-2x)+$$

$$x^2\phi_5(y-2x)+\phi_6(y+3x)+\phi_7(y-4x)$$

Ex. 4:- Find the general solution of

$$(D_x^2+D_y^2)Z=0$$

Solution:-

$$m^2+1=0 \Rightarrow m^2=-1 \Rightarrow m=\mp i$$

$$m_1=+i$$

$$m_2=-i$$

$$Z=\phi_1(y+m_1x)+\phi_2(y+m_2x)$$

$$Z=\phi_1(y+ix)+\phi_2(y-ix)$$

Ex. 5:- Find the general solution of

$$(D_x^2-2D_xD_y+5D_y^2)Z=0$$

Solution

$$m^2-2m+5=0$$

$$a=1,\ b=-2,\ c=5$$

$$m = \frac{-b \mp \sqrt{b^2-4ac}}{2a} = \frac{2 \mp \sqrt{4-20}}{2} = \frac{2 \mp \sqrt{-16}}{2}$$

$$m = 1 \mp 2i$$

$$m_1 = 1+2i$$

$$m_2 = 1-2i$$

$$Z = \phi_1(y+m_1x) + \phi_2(y+m_2x)$$

$$Z = \phi_1(y+(1+2i)x) + \phi_2(y+(1-2i)x)$$

Ex. 6 :- Find the general solution of

$$(D_x^4 - D_x^3D_y + 2D_x^2D_y^2 - 5D_xD_y^3 + 3D_y^4)Z = 0$$

solution:-

$$m^4 - m^3 + 2m^2 - 5m + 3 = 0$$

$$m^3(m-1) + 2m^2 - 2m - 3m + 3 = 0$$

$$m^3(m-1) + 2m(m-1) - 3(m-1) = 0$$

$$(m-1)(m^3+2m-3) = 0$$

$$\begin{array}{r|l} & m^2+m+3 \\ \hline m-1 & m^3+2m-3 \\ - & m^3-m^2 \\ \hline & m^2+2m-3 \\ - & m^2-m \\ \hline & 3m-3 \\ - & 3m-3 \\ \hline & 0 \end{array}$$

$$m = 1$$

$m-1=0$ and m^2+m+3

$$(m-1)^2(m^2+m+3) = 0$$

$a=1,\ b=1,\ c=3$

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow m = \frac{-1 \mp \sqrt{1-12}}{2} \Rightarrow m = -\frac{1}{2} \mp \frac{\sqrt{11}}{2} i$$

$$m_1 = m_2 = 1 \quad , m_3 = -\frac{1}{2} + \frac{\sqrt{11}}{2} i \quad , m_4 = -\frac{1}{2} - \frac{\sqrt{11}}{2} i$$

$$\boxed{\begin{aligned} Z = {} & \phi_1(y+x) + x\phi_2(y+x) + \phi_3\left(y + \left(-\tfrac{1}{2} + \tfrac{\sqrt{11}}{2} i\right)x\right) + \\ & \phi_4\left(y + \left(-\tfrac{1}{2} - \tfrac{\sqrt{11}}{2} i\right)x\right) \end{aligned}}$$

$$F(D_x, D_y)Z = f(x,y)$$

general solution — Particular solution

1) $f(x,y) = 0$

2) $f(x,y) \neq 0$

استخراج الحل الخاص ((Particular solution))

أولاً) طريقة الفرضيات

(تعتمد على لاكرانج)

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

ثانياً) الحالات الخمسة

case 1 ; e^{ax+by}

Case 2 ; $\sin(ax+by)$

or $\cos ..$

case 3 ; $x^a \cdot y^b$

case 4 ; $e^{ax+by} \cdot V$ where V is function of x and y.

case 5 ; $g(ax+by)$ where $F(a,b) \neq 0$

case 6 ; $g(ax+by)$ where $F(a,b) = 0$

أولاً) طريقة الفرضيات (تعتمد على لاكرانج)

Ex. 1:- Solve $(D_x^2 - D_y^2)Z = \sec^2(x+y)$

solution:-

Find general solution !! $f(x,y) = 0$

$$(D_x^2 - D_y^2)Z = 0$$

$$\Rightarrow m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m_1 = 1 \ , \ m_2 = -1$$

$$\boxed{Z_g = \Phi_1(y+x) + \Phi_2(y-x)}$$ general solution

Find Particular solution !! $f(x,y) \neq 0$

$$Z = \frac{1}{D_x^2 - D_y^2}\sec^2(x+y)$$

$$\Rightarrow Z = \frac{1}{(D_x - D_y)}\underbrace{\frac{1}{(D_x + D_y)}\sec^2(x+y)}_{u_1} \quad \text{....(1)}$$

let $u_1 = \dfrac{1}{D_x + D_y}\sec^2(x+y)$

$(D_x + D_y)u_1 = \sec^2(x+y)$ لاكرانج

$$\frac{dx}{1}=\frac{dy}{1}=\frac{du_1}{\sec^2(x+y)}$$

$$\int dx=\int dy \Rightarrow x=y+a \Rightarrow a=x-y$$

$$\rightarrow y=x-a$$

$$dx=\frac{du_1}{\sec^2(x+y)} \Rightarrow \int \frac{1}{2}\sec^2(2x-a)\,2dx=\int du_1$$

$$\Rightarrow \frac{1}{2}\tan(2x-a)=u_1+b$$

let $b=0$ and replacing a

$$u_1=\frac{1}{2}\tan(x+y) \quad \text{in}(1)$$

$$z=\frac{1}{D_x-D_y}\cdot\frac{1}{2}\tan(x+y)$$

$$\Rightarrow (D_x-D_y)z=\frac{1}{2}\tan(x+y)$$ ((لاكمال نج))

$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{\frac{1}{2}\tan(x+y)}$$

$$\int dx=-\int dy \Rightarrow x+y=a$$

$$dx = \frac{dz}{\frac{1}{2}\tan(x+y)} \Rightarrow \int \frac{1}{2}\tan(a)\,dx = \int dz$$

$$\Rightarrow \frac{1}{2}x\tan(a) = z + b$$

let $b=0$ and replacing a

$$\boxed{z_p = \frac{1}{2}x\tan(x+y)}$$ Particular solution

$$z = z_g + z_p$$

$$\boxed{z = \phi_1(y+x) + \phi_2(y-x) + \frac{1}{2}x\tan(x+y)}$$

general solution