الطريقة رقم 4 :- اذا كان شكل المعادلة $f(p,q,z)=0$

طريقة الحل

1. نفرض $u=x+ay$

2. نستبدل كل من $p=\frac{dz}{du}$ و $q=a\frac{dz}{du}$ في المعادلة الأصلية

$$p=\frac{\partial z}{\partial x}\cdot\frac{\partial u}{\partial u}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}=\frac{dz}{du}$$

$$q=\frac{\partial z}{\partial y}\cdot\frac{\partial u}{\partial u}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}=a\frac{dz}{du}$$

3. نحل المعادلة الناتجة ثم نعيد قيمة $u=x+ay$ لينتج الحل التام

Ex. 1 :- Solve $z=p+q$

Solution

Let $u=x+ay$ and $p=\frac{dz}{du}$, $q=a\frac{dz}{du}$

$$\Rightarrow z=\frac{dz}{du}+a\frac{dz}{du}\Rightarrow z=(1+a)\frac{dz}{du}$$

$$\Rightarrow\int du=(1+a)\int\frac{dz}{z}\Rightarrow u=(1+a)\ln z+c$$

$$\Rightarrow x+ay=(1+a)\ln z+c$$

$$\Rightarrow \ln z^{(1+a)} = x+ay-c$$

$$\Rightarrow z^{(1+a)} = e^{x+ay-c}$$

$$\Rightarrow z = e^{\frac{x+ay-c}{1+a}} \longrightarrow \text{complete solution}$$

Ex. 2 :- Solve $p^2 z - q^2 = 1$

Solution

Let $u = x+ay$ and $p=\frac{dz}{du}$, $q=a\frac{dz}{du}$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 z - a^2\left(\frac{dz}{du}\right)^2 = 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 (z-a^2) = 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{1}{z-a^2} \Rightarrow \frac{dz}{du} = \frac{1}{\mp\sqrt{z-a^2}}$$

$$\Rightarrow \int \mp (z-a^2)^{\frac{1}{2}} dz = \int du$$

$$\Rightarrow \mp \frac{2}{3}(z-a^2)^{\frac{3}{2}} = u + c$$

$$\Rightarrow \mp \frac{2}{3}(z-a^2)^{\frac{3}{2}} = x + ay + c \rightarrow \text{complete solution}$$

الطريقة رقم 5 :- اذا كانت المعادلة $f(x,p)=g(y,q)=0$

طريقة الحل :-

1. نعزل p و x في طرف و q و y في الطرف الآخر.
2. نساوي كل من $f(x,p)=a$ و $g(y,q)=a$ لنستخرج قيمة p بدلالة a و q بدلالة a.
3. نعوض قيمة p و q في $dz = p\,dx + q\,dy$ ونكامل كل الاطراف لإستخراج الحل التام

Ex.1 :- Solve $p = 2xq^2$

Solution

$$\frac{p}{x} = 2q^2$$

$$\frac{p}{x} = a \Rightarrow \boxed{p = ax}$$

$$2q^2 = a \Rightarrow \boxed{q = \mp\sqrt{\frac{a}{2}}}$$

$$dz = p\,dx + q\,dy$$

$$\Rightarrow \int dz = \int ax\,dx + \int \left(\mp\sqrt{\tfrac{a}{2}}\right) dy$$

$$\Rightarrow z = \frac{ax^2}{2} \mp \sqrt{\frac{a}{2}}\, y + C \rightarrow \text{complete solution}$$

Ex.2 :- Solve $xq - y^2 p - x^2 y^2 = 0$

Solution

$$xq = y^2(p - x^2) \Rightarrow \frac{q}{y^2} = \frac{p - x^2}{x}$$

$$\frac{q}{y^2} = a \Rightarrow q = ay^2$$

$$\frac{p - x^2}{x} = a \Rightarrow p - x^2 = ax \Rightarrow p = ax + x^2$$

$$dz = p\,dx + q\,dy$$

$$\Rightarrow \int dz = \int (ax + x^2)\,dx + \int ay^2\,dy$$

$$\Rightarrow z = \frac{ax^2}{2} + \frac{x^3}{3} + \frac{ay^3}{3} + C \rightarrow \text{complete solution}$$

Ex.3 :- Solve $p - 3x^2 = q^2 - y$

Solution

$$p - 3x^2 = a \Rightarrow p = a + 3x^2$$

$$q^2 - y = a \Rightarrow q^2 = a + y \Rightarrow q = \mp\sqrt{a+y}$$

$$dz = p\,dx + q\,dy$$

$$\Rightarrow \int dz = \int (a + 3x^2)\,dx + \int \left(\mp\sqrt{a+y}\right) dy$$

$$\Rightarrow z = ax + x^3 \mp \frac{2}{3}(a+y)^{\frac{3}{2}} + C \rightarrow \text{Complete soluti}$$

الطريقة رقم 6 :- ((طريقة جاربت)) تحل هذه الطريقة

المعادلات الجزئية الغير خطية التي تكون

من الرتبة الاولى و أي درجة

$$f(z, p, q, x, y) = 0$$

طريقة الحل :-

1. نصفر المعادلة.

2. نجهز كل ما يلي f_z ، f_p ، f_q ، f_x ، f_y من المعادلة.

3. نستعمل معادلة جاربت المساعدة :-

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

ثم نأخذ أي كسرين ونستخرج قيمة p او q ثم نعوض قيمة المستخرج في المعادلة الاصلية لإستخراج الآخر.

4. نعوض p و q في $dz = p\,dx + q\,dy$ لإستخراج الحل التام.

Ex. 1 :- Solve $px + qy = pq$ ..... (*)

Solution :-

$$px + qy - pq = 0$$

$$f_p = x - q \ , \ f_q = y - p \ , \ f_z = 0 \ , \ f_x = p \ , \ f_y = q$$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{p + p(0)} = \frac{dq}{q + q(0)} = \frac{dz}{-px + pq - qy + pq} = \frac{dx}{q - x} = \frac{dy}{p - y}$$

$$\int \frac{dp}{p} = \int \frac{dq}{q} \Rightarrow \ln p = \ln q + \ln a$$

$$\Rightarrow \ln p = \ln aq$$

$$\Rightarrow p = aq \ ...... (1) \text{ in } (*)$$

$$[aqx + qy = aq^2] \div q$$

$$ax + y = aq \Rightarrow q = \frac{ax + y}{a} \text{ in } (1)$$

$$p = a\left(\frac{ax + y}{a}\right) \Rightarrow p = ax + y$$

$$dz = p\,dx + q\,dy$$

$$\int dz = \int ax\,dx + \int d(xy) + \int \frac{y}{a}\,dy$$

$$z = \frac{a}{2}x^2 + xy + \frac{y^2}{2a} + c \rightarrow \text{complete solution}$$

Ex. 2 :- Solve $2zx - px^2 - 2qxy + pq = 0$ ....⊛

Solution

$$f_p = -x^2 + q \ , \ f_q = -2xy + p \ , \ f_z = 2x$$

$$f_x = 2z - 2px - 2qy \ , \ f_y = -2qx$$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{2z - \cancel{2px} + \cancel{2xp} - 2qy} = \frac{dq}{\underbrace{-\cancel{2qx} + \cancel{2xq}}_{0}} = \frac{dz}{+px^2 - pq + 2qxy - pq} =$$

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

م/ دائماً الكسر الذي يصفر مقامه مباشرة نساوي البسط بالصفر

$$\int dq = \int 0 \Rightarrow q = a \quad \text{in} \circledast$$

$$2zx - px^2 - 2axy + ap = 0$$

$$\Rightarrow px^2 - ap = 2zx - 2axy$$

$$\Rightarrow p(x^2 - a) = 2zx - 2axy$$

$$\Rightarrow p = \frac{2x(z - ay)}{x^2 - a}$$

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{2x(z - ay)}{x^2 - a}dx + a\,dy$$

$$\Rightarrow dz - a\,dy = \frac{2x(z - ay)}{x^2 - a}dx$$

$$\Rightarrow \frac{dz - a\,dy}{z - ay} = \frac{2x}{x^2 - a}dx$$

$$\Rightarrow \ln(z - ay) = \ln(x^2 - a) + \ln c$$

$$\Rightarrow \ln(z - ay) = \ln(x^2 - a)c$$

$$z - ay = (x^2 - a)c$$

$$z = (x^2 - a)c + ay \rightarrow \text{complete solution}$$

Ex.3 :- Solve $z = px + qy + p^2 + q^2$

Solution

$$z - px - qy - p^2 - q^2 = 0$$

$$f_p = -x - 2p \,,\; f_q = -y - 2q \,,\; f_z = 1$$

$$f_x = -p \,,\; f_y = -q$$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{\cancel{-p} + \cancel{p}} = \frac{dq}{\cancel{-q} + \cancel{q}} = \frac{dz}{xp + 2p^2 + yq + 2q^2} = \frac{dx}{x + 2p} = \frac{dy}{y + 2q}$$

$$\int dp = \int 0 \Rightarrow p = a$$

$$\int dq = \int 0 \Rightarrow q = b$$

$$dz = p\,dx + q\,dy$$
$$\int dz = \int a\,dx + \int b\,dy$$

$$z = ax + by + C \rightarrow \text{complete solution}$$

Exercises

Solve the following equations :-

1- $q = 3p^2$

<u>solution</u> :-

$$p = a \ , \ q = b$$

$$\Rightarrow \ b = 3a^2$$

from $z = ax + by + c$

$$z = ax + 3a^2 y + C \rightarrow \text{complete solution}$$

$2-\ zpq = p+q$

Solution:-

$$u = x+ay$$

$$p = \frac{dz}{du}\ ,\ q = a\frac{dz}{du}$$

$$z\left(\frac{dz}{du}\right)\left(a\frac{dz}{du}\right) = \frac{dz}{du} + a\frac{dz}{du}$$

$$\Rightarrow az\left(\frac{dz}{du}\right)^2 = \frac{dz}{du}(1+a)$$

$$\Rightarrow az\frac{dz}{du} = 1+a \quad \Rightarrow \int az\,dz = (1+a)\int du$$

$$\Rightarrow \frac{a}{2}z^2 = (1+a)u + C$$

$$\Rightarrow \frac{a}{2}z^2 = (1+a)(x+ay) + C \longrightarrow \text{complete solution}$$

$$3- \quad p^2 - y^2 q = y^2 - x^2 \quad \ldots\ldots (*)$$

Solution :-

$$p^2 + x^2 = y^2 q + y^2$$

$$p^2 + x^2 = a \Rightarrow p^2 = a - x^2$$

$$\Rightarrow p = \mp\sqrt{a - x^2}$$

$$y^2 q + y^2 = a \Rightarrow y^2 q = a - y^2$$

$$\Rightarrow q = \frac{a}{y^2} - 1$$

$$dz = p\,dx + q\,dy$$

$$\Rightarrow \int dz = \int \mp\sqrt{a^2 - x^2}\,dx + \int a y^{-2}\,dy - \int dy \qquad \boxed{a = a^2}$$

$$\sqrt{a^2 - x^2}\,dx \qquad \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

$$x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$$

$$\sqrt{a^2 - a^2\sin^2\theta}\quad a\cos\theta\,d\theta$$

$$a\sqrt{1 - \sin^2\theta}\quad a\cos\theta\,d\theta$$

$a^2\sqrt{\cos^2\theta}\ \cos\theta\, d\theta$

$a^2\int \cos^2\theta\, d\theta \Rightarrow a^2\int\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$

$\Rightarrow a^2\left(\frac{1}{2}\theta - \sin 2\theta\right)$

$\Rightarrow \int dz = \int \mp\sqrt{a^2-x^2}\, dx + \int a y^{-2} dy - \int dy$

$\Rightarrow z = \mp\left(a^2\left(\frac{1}{2}\theta - \sin 2\theta\right)\right) - \frac{a}{y} - y + c$