Def: If $(I,+,\cdot)$ is an ideal of the ring $(R,+,\cdot)$ then the natural mapping $nat_I : R \longrightarrow R/I$ given by $nat_I(a) = a+I$, $\forall a \in R$

Theorem 3-12: Let $(I,+,\cdot)$ be an ideal of the ring $(R,+,\cdot)$ the natural map is a homomorphisem from a ring $(R,+,\cdot)$ onto the quotient ring $(R/I,+,\cdot)$ with kernel equal to I.

proof: first T.P. $nat_I : R \longrightarrow R/I$ is homo.

let $a,b \in R$

1) $nat_I(a+b) = (a+b)+I = (a+I)+(b+I) = nat_I(a)+nat_I(b)$

2) $nat_I(ab) = ab+I = (a+I)(b+I) = (nat_I(a))(nat_I(b))$

$\therefore$ nat_I is homo.

Now, to prove nat_I is onto

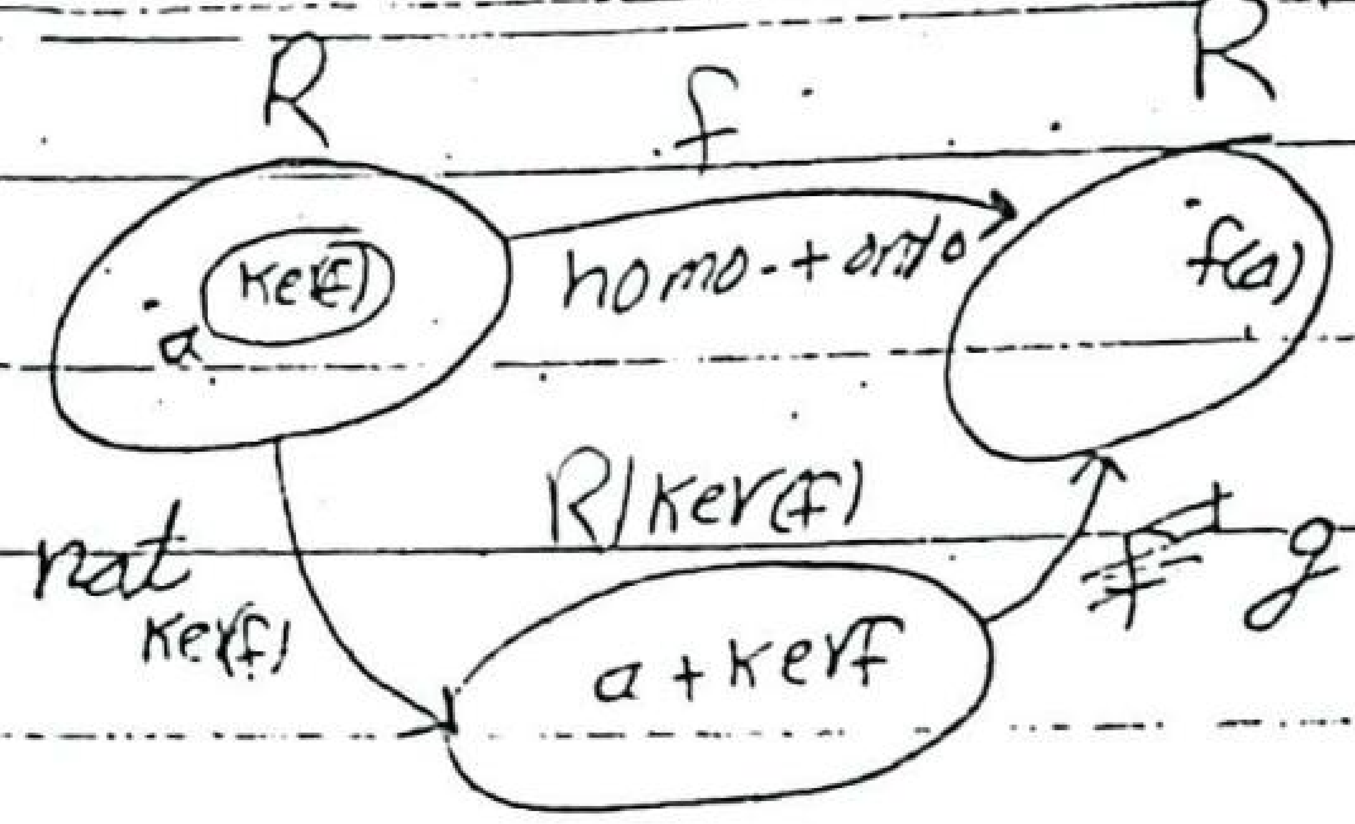
$\forall\ a+I \in R/I,\ a \in R \Rightarrow \exists a \in R$ s.t. $nat_I(a) = a+I$

$\therefore$ nat_I is onto.

$$\begin{aligned} Ker(nat_I) &= \{a \in R : nat_I(a) = 0+I\} \\ &= \{a \in R : a+I = I\} \\ &= \{a \in R : a \in I\} \\ &= I. \end{aligned}$$

Theorem 3-13 (Fundamental theorem)

let $f : (R,+,\cdot) \longrightarrow (R',+',\cdot')$ be a ring homo. from a ring R onto R' then $R/ker(f) \cong R'$.



Proof: Let $g : R/\ker(f) \longrightarrow R'$ defined by

$$g(a+\ker(f)) = f(a) \quad , \forall\, a+\ker(f) \in R/\ker(f)$$

To show that g is well defined.

Let $a+\ker(f),\ b+\ker(f) \in R/\ker(f)$ s.t. $g(a+\ker(f)) = g(b+\ker(f))$

$$a+\ker(f) = b+\ker(f)$$

$$\Rightarrow a-b \in \ker(f)$$

$$\therefore f(a-b) = 0$$

$$f(a)-f(b) = 0$$

$$\therefore f(a) = f(b)$$

$\therefore$ g is well defined.

Now, T.P. g is homo.

Let $a+\ker(f),\ b+\ker(f) \in R/\ker(f)$

1) $g(a+\ker(f)+b+\ker(f)) = g(a+b+\ker(f)) = f(a+b)$

$= f(a)+f(b)$ [f is homo.]

$= g(a+\ker(f)) + g(a+\ker(f))$

2) $g((a+\ker(f))(b+\ker(f))) = g(ab+\ker(f)) = f(a \cdot b)$.

$= f(a) \cdot f(b)$ [f is homo.]

$= g(a+\ker(f)) \cdot g(b+\ker(f))$

$\therefore$ g is homo.

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To show g is onto

let $x \in R' \Rightarrow \exists\, a \in R$ s.t. $x = f(a)$

$a \in R \Rightarrow a + \ker(f) \in R/\ker(f) \Rightarrow f(a) = g(a+k)$

$\Rightarrow x = g(a+k)$

$\therefore g$ is onto

T.P. g is 1-1

let $a+\ker(f),\ b+\ker(f) \in R/\ker(f)$ s.t.

$g(a+\ker(f)) = g(b+\ker(f))$

$\Rightarrow f(a) = f(b)$

$\Rightarrow f(a) - f(b) = 0$

$\Rightarrow f(a-b) = 0$

$\Rightarrow a - b \in \ker(f)$

$\Rightarrow a + \ker(f) = b + \ker(f)$

$\therefore g$ is 1-1.

$\therefore R/\ker(f) \cong R'$

Ex.: Show that $\mathbb{Z}_4/\{0,2\} \simeq \mathbb{Z}_2$.

Sol.: we must find a function

$f: \mathbb{Z}_4 \to \mathbb{Z}_2$ s.t. f is homo. onto & $\ker(f) = \{0,2\}$

let $f: \mathbb{Z}_4 \to \mathbb{Z}_2$ given by

$f(0) = f(2) = 0$

$f(1) = f(3) = 1$

$\Rightarrow f$ is onto & homo.

$\ker(f) = \{0,2\}$

$\therefore \mathbb{Z}_4/\{0,2\} \simeq \mathbb{Z}_2$

Ex-: IS $\mathbb{Z}/\mathbb{Z}_e \simeq \mathbb{Z}$ where $f: \mathbb{Z} \to \mathbb{Z}$ s.t. $f(n)=n$.

~~sol: f is homo & onto~~

~~but $Ker(f)=\{0\} \neq \mathbb{Z}_e$~~

~~$\therefore \mathbb{Z}/\mathbb{Z}_e \not\simeq \mathbb{Z}$~~

Ex-: Show that $\mathbb{Z}_{32}/\{0,8,16,24\} \simeq \mathbb{Z}_8$

sol-: To find a homo. $f: \mathbb{Z}_{32} \to \mathbb{Z}_8$ which is onto and $Ker(f)=\{0,8,16,24\}$

suppose that

$$f(1)=f(9)=f(17)=f(25)=\cancel{f(33)}=1$$
$$f(2)=f(10)=f(18)=f(26)=2$$
$$f(3)=f(11)=f(19)=f(27)=3$$
$$f(4)=f(12)=f(20)=f(28)=4$$
$$f(5)=f(13)=f(21)=f(29)=5$$
$$f(6)=f(14)=f(22)=f(30)=6$$
$$f(7)=f(15)=f(23)=f(31)=7$$
$$f(8)=f(16)=f(24)=f(32)=0$$

f is homo. and onto and

$Ker(f) = \{a \in \mathbb{Z}_{32} ; f(a)=0\}$

$= \{a \in \mathbb{Z}_{32} ; a=8,16,24,32\}$

$= \{8,16,24,32\}$

$\therefore \mathbb{Z}_{32}/\{8,16,24,0\} \simeq \mathbb{Z}_8$

or $f(a)=a_{\mathbb{Z}_8}$, f is homo. & onto

$Ker.f = \{a \in \mathbb{Z}_{32} ; f(a)=0_{\mathbb{Z}_8}\}$

$= \{a \in \mathbb{Z}_{32} ; a=0,8,16,24\}$

$\mathbb{Z}_{32}/\{0,8,16,24\} \simeq \mathbb{Z}_8$