$\because 0 \in R$ and $f(0) = 0'$

$\because f$ is one-one

$\therefore \forall a \in R$ s.t. $a \neq 0 \Rightarrow f(a) \neq 0'$

let $a', b' \neq 0'$, $a', b' \in R'$

$\because f$ is 1-1 and onto

$\therefore \exists!\ a, b \in R$ s.t. $a' = f(a)$, $b' = f(b)$

$\because (R, +, \cdot)$ without zero divisors and $a \neq 0, b \neq 0$

$\Rightarrow a \cdot b \neq 0$

$\Rightarrow f(a \cdot b) \neq f(0)$ [f is 1-1]

$\Rightarrow f(a) \cdot f(b) \neq 0'$ [f is homo. f: $f(0) = 0'$]

$\Rightarrow a' \cdot b' \neq 0'$

$\therefore (R', +', \cdot')$ is without zero divisors.

Def.: A ring $(R, +, \cdot)$ is said to be a skew-field or a division ring if R is a ring with identity in which every non-zero element has a multiplicative inverse.

Ex.: The ring $(M_2(\mathbb{R}), +, \cdot)$ is a skew-field.
where $M_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$

Note: division ring — حلقة القسمة

Qusiont ring — حلقة القسمة

Theorem 3-11: Every isomorphic image of a division ring (skew-field) is a division ring (skew-field).

proof: let $f:(R,+,\cdot)\to(R',+',\cdot')$ is an isomorphisem of a ring R (a division ring) onto a ring R' and let $(R,+,\cdot)$ be a skew-field.

T.P. $(R',+',\cdot')$ is a skew-field (division ring).

$\because$ 1 is the identity element of R and by theorem (3-7)(ii) then $f(1)=1'$ is the identity element of R'.

let $0' \neq a' \in R'$

$\because$ f is onto $\Rightarrow \exists\ 0\neq a\in R$ s.t. $f(a)=a'$

$\because a\in R$ and R is a skew-field (division ring)

$\therefore \exists\ a^{-1}\in R$ s.t. $a\cdot a^{-1}=a^{-1}\cdot a=1$

$\Rightarrow\ f(a\cdot a^{-1})=f(a^{-1}\cdot a)=f(1)$

$\Rightarrow\ f(a)\cdot f(a^{-1})=f(a^{-1})\cdot f(a)=1'$ [f is homo. + $f(1)=1'$]

$\Rightarrow\ a'\cdot f(a)^{-1}=f(a)^{-1}\cdot a'=1'$

$\therefore (a')^{-1}=f(a)^{-1}\in R'$

$\therefore$ Every non-zero element of R' has mult. inverse.

$\therefore (R',+',\cdot')$ is a skew-field.

corollary: Every isomorphic image of a field is a field.

proof: let $f:(R,+,\cdot)\to(R',+',\cdot')$ be an isomorphism from a ring $(R,+,\cdot)$ into a ring $(R',+',\cdot')$

and $(R,+,\cdot)$ is a field
T.P $(R',+',\cdot')$ is a field
by theorem 3.11 $\Rightarrow$ R' is a skew-field
and by theorem (3.7)(i) $\Rightarrow$ R' is a commu. ring.
$\therefore$ $(R',+',\cdot')$ is a field.

Def.: let $(R,+,\cdot)$ be a ring and $(A,+,\cdot)$ be a subring of R the function $f: A \longrightarrow R$ s.t. $f(x)=x, \forall x \in A$ is called inclusion homo. (تعريف/الاحتواء) iff f is monomorphism (homo. + 1-1).

Ex.: let $(\mathbb{R},+,\cdot)$ be a ring and $(\mathbb{Z},+,\cdot)$ a subring of $\mathbb{R}$ then a function $f: \mathbb{Z} \longrightarrow \mathbb{R}$ s.t. $f(n)=n$ is inclusion homo.

sol.: $\because$ $(\mathbb{Z},+,\cdot)$ is a subring of $(\mathbb{R},+,\cdot)$
and f is homo. since $\forall a,b \in \mathbb{Z}$

1) $f(a+b) = a+b = f(a)+f(b)$

2) $f(a\cdot b) = a\cdot b = f(a)\cdot f(b)$

and since f is 1-1

let $n,m \in \mathbb{Z}$ s.t.

$f(n) = f(m)$

$\Rightarrow$ $n=m$

$\therefore$ f is 1-1

$\therefore$ f is inclusion homo.