Theorm 3.8 :- If $f : (R, +, \cdot) \longrightarrow (\acute{R}, \acute{+}, \acute{\cdot})$ is ahomomorphisem Then $(\ker.f, +, \cdot)$ is an ideal of R.

proof :- Let $a, b \in \ker.f \Rightarrow f(a) = \acute{0}$ and $f(b) = \acute{0}$

i) T.P. $a-b \in \ker.f$

$$f(a-b) = f(a+(-b)) = f(a) + f(-b)$$
$$= f(a) - f(b)$$
$$= \acute{0} - \acute{0}$$
$$= \acute{0}$$
$$\Rightarrow a-b \in \ker.f.$$

ii) T.P $ar, ra \in \ker f$.

$$f(a \cdot r) = f(a) \cdot f(r)$$
$$= \acute{0} \cdot f(r)$$
$$= \acute{0}$$
$$\Rightarrow ar \in \ker.f$$

$$f(r \cdot a) = f(r) \cdot f(a)$$
$$= f(r) \cdot \acute{0}$$
$$= \acute{0}$$
$$\Rightarrow ra \in \ker.f.$$

$\therefore (\ker.f, +, \cdot)$ is an ideal of $(R, +, \cdot)$

Theorm 3.9 :- If $f:(R,+,\cdot) \longrightarrow (\acute{R},\acute{+},\acute{\cdot})$ is ahomomorphisem then f is ~~onto~~ one to one iff

$$Ker.f = \{0\}.$$

Proof :- suppose that f is 1-1

T.P $Ker\ f = \{0\}$

$(\Rightarrow)$ let a $a \in Ker.f \Longrightarrow f(a) = \acute{0}$

$\because f(0) = \acute{0}$

$\therefore f(a) = f(0)$

$\because f$ is 1-1

$\therefore a = 0$

$\therefore Ker.f \subseteq \{0\}$

$\because f(0) = \acute{0}$

$\therefore 0 \in Ker.f$

$\therefore \{0\} \subseteq Ker.f$

$\therefore Ker\ f = \{0\}$

$(\Leftarrow)$ suppose that $Ker.f = \{0\}$

T.P f is 1-1.

Let $a, b \in R$ s.t $f(a) = f(b)$

$$f(a) - f(b) = \acute{0}$$

$$f(a-b) = \acute{0}$$

$$\therefore \quad a - b = 0$$

$$\Rightarrow a = b$$

$\therefore f$ is 1-1

Def :- Let $f : (R, +, \cdot) \longrightarrow (\acute{R}, \acute{+}, \acute{\cdot})$ be a homo-

Then the image of F is $Im \cdot f = \{ \acute{a} \in \acute{R} : \exists a \in R$ s.t $f(a) = \acute{a} \}$.

Remark :- If $f : (R, +, \cdot) \longrightarrow (\acute{R}, \acute{+}, \acute{\cdot})$ is a homo-

Then $Im \cdot f$ is subring by Theorm (3.5), but not ideal.

Ex :- Let $f : \mathbb{Z} \longrightarrow R$ s.t $f(n) = n$, $\forall n \in \mathbb{Z}$

f is homo. since

① $f(n+m) = n+m = f(n) + f(m)$

② $f(n \cdot m) = n \cdot m = f(n) \cdot f(m)$.

$Im \cdot f = \mathbb{Z}$

* $Im \cdot f = \mathbb{Z}$

$\therefore$ Im.f is a subring but $\mathbb{Z}$ is not ideal

since $2 \in \mathbb{Z}, \frac{1}{3} \in R \Rightarrow 2 \cdot \frac{1}{3} = \frac{2}{3} \notin \mathbb{Z}$

and $\frac{1}{3}(2) = \frac{2}{3} \in \mathbb{Z}$

$\therefore (Im \cdot f, \acute{+}, \acute{\cdot})$ is subring but not ideal.

Def :- Let $f : (R, +, \cdot) \longrightarrow (\acute{R}, \acute{+}, \acute{\cdot})$ be a homo- from a ring $(R, +, \cdot)$ into a ring $(\acute{R}, \acute{+}, \acute{\cdot})$ Then

① homo + one - one $\Longrightarrow$ menomorphisem.

② homo + onto $\Longrightarrow$ Epimorphisem.

③ homo + bijective $\Longrightarrow$ Isomorphisem,

④ homo + $R = \acute{R}$ $\Longrightarrow$ Endomorphisem.

⑤ Isomor + $R = \acute{R}$ $\Longrightarrow$ automorphisem.

EX :- A function $f : (\mathbb{Z}, +, \cdot) \longrightarrow (R, +, \cdot)$ s.t $f(n) = n$ is menomorphisem.

EX:- A function $g:(Z,+,\cdot) \longrightarrow (Z,+,\cdot)$ s.t

$g(a)=a$ is automorphisem.

Ex:- A function $h:(R,+,\cdot) \longrightarrow (R,+,\cdot)$ s.t $f(x)=0$

$\forall x \in R$, Then h is Endomorphisem.

not:- isomor denoted by $\simeq$

Therm 3-10:- Every isomorphic image of aring without Zero divisors is aring without Zero divisor.

Proof:- let $f:(R,+,\cdot) \longrightarrow (\acute{R},\acute{+},\acute{\cdot})$ be isomorphisem and $(R,+,\cdot)$ is aring without Zero divisor.

T.P $(\acute{R},\acute{+},\acute{\cdot})$ is aring without Zero divisors,

$\because 0 \in R$ and $f(0)=\acute{0}$

$\because f$ is 1-1

$\therefore \forall a \in R$ s.t $a \neq 0 \Rightarrow f(a) \neq \acute{0}$

Let $\acute{a},\acute{b} \neq \acute{0}$, $\acute{a},\acute{b} \in \acute{R}$

$\because f$ is 1-1 and onto

$\therefore \exists\ a, b \in R$ s.t $a' = f(a)$, $b' = f(b)$

$\because (R, +, \cdot)$ without zero divisors and $a \neq 0, b \neq 0$

$\Rightarrow a \cdot b \neq 0$

$\Rightarrow f(a \cdot b) \neq f(0)$ [f is 1-1]

$\Rightarrow f(a) \cdot f(b) \neq 0'$ [f is hom- and $f(0) = 0'$]

$\Rightarrow a' \cdot b' \neq 0'$

$\therefore (R', +', \cdot')$ is without zero divisors.

<u>Def</u> :- A ring $(R, +, \cdot)$ is said to be a <u>skew-field</u> or a <u>division ring</u> of R is a ring with identity in which every non-zero element has a multiplicative inverse.