Theorm 3.5 :- If $F(R,+,\cdot) \longrightarrow (R',+',\cdot')$ is homomorphisem Then $F(R)$ is asubring of R'

~~Proof~~

Theorm 3.6 :- If $f : (R,+,\cdot) \longrightarrow (R',+',\cdot')$ be aring homomorphisim and

① I is anideal of R Then $f(I)$ is anideal of R' where f is onto.

② If I' is anideal of R' then $f^{-1}(I')$ is an ideal of R.

proof :- ① By theorm 3.3 $\Rightarrow f(I)$ is asubring

let $a \in f(I) \Rightarrow \exists x \in I$ s.t $a = f(x)$, $r \in R' \Rightarrow \exists y \in R$ s.t $f(y) = r$, since f is onto.

$a \cdot r = f(x) \cdot f(y) = f(x \cdot y)$

$\because x \in I$, $y \in R$, and I is anideal $\Rightarrow xy \in I$

$\Rightarrow f(xy) \in f(I)$

$\Rightarrow a \cdot r \in f(I)$

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$ra = f(y) f(x) = f(xy)$

$\because x \in I, y \in R$ and I is an ideal $\Rightarrow yx \in I$

$\Rightarrow f(yx) \in f(I)$

$ra \in f(I)$

$\therefore (f(I), \acute{+}, \acute{\cdot})$ is an ideal of $(\acute{R}, \acute{+}, \acute{\cdot})$

② $f^{-1}(\acute{I}) = \{a \in f^{-1}(\acute{I}) : f(a) \in \acute{I}\}$

let $a \in f^{-1}(\acute{I}), b \in R$

$f(a \cdot b) = f(a) \cdot f(b)$

$\because a \in f^{-1}(\acute{I}) \Rightarrow f(a) \in \acute{I}, b \in R \Rightarrow f(b) \in \acute{R}$

and $\acute{I}$ is an ideal $\Rightarrow f(a) \cdot f(b) \in \acute{I}$

$\therefore a \cdot b \in f^{-1}(\acute{I})$

$f(b \cdot a) = f(b) f(a)$

$\because a \in f^{-1}(\acute{I}) \Rightarrow f(a) \in \acute{I}$ and $f(b) \in \acute{R}$ and

$\acute{I}$ is an ideal of $\acute{R} \Rightarrow f(b) \cdot f(a) \in \acute{I}$

$\Rightarrow b \cdot a \in f^{-1}(\acute{I})$

$\therefore (f^{-1}(\acute{I}), +, \cdot)$ is an ideal of R.

Therm 3.7 : Every onto homomorphisem Image of

① a commutative ring is a commutative ringe.

② a ring with identity is a ring with identity.

Proof : Let $f : (R, +, \cdot) \longrightarrow (R', +', \cdot')$ be a homomorphisem from a ring $(R, +, \cdot)$ onto a ring $(R', +', \cdot')$

① let $(R, +, \cdot)$ be a comm - ring.

T.P $(R', +', \cdot')$ is a comm - ring.
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Let $a, b \in R'$

$\because$ f is onto

$\therefore \exists\, x, y \in R$ s.t $a = f(x)$, $b = f(y)$

$a \cdot b = f(x) \cdot f(y)$

$= f(x \cdot y) = f(y \cdot x)$ [since R is comm.]

$= f(y) \cdot f(x)$ [f is hom.]

$= b \cdot a$

$\therefore R'$ is a comm - ring.

② Let R be a ring with identity.

and $a \in R'$ since f is onto Therefore

$\exists x \in R$ s.t $f(x) = a$

$\because x \cdot 1 = 1 \cdot x = x$

$f(x \cdot 1) = f(1 \cdot x) = f(x)$

$f(x) \cdot f(1) = f(1) \cdot f(x) = f(x)$ · [f is homo.]

$a \cdot f(1) = f(1) \cdot a = a$

$\therefore f(1)$ is the identity element of R'.

<u>Def</u>:- Let f be a homomorphisem of a ring $(R, +, \cdot)$ into a ring $(R', +', \cdot')$ Then the <u>Kernel</u> of f is defind as

$Ker.f = \{ a \in R : f(a) = 0'$, where $0'$ denotes the zero of $R' \}$.

Ex :- let $f(\mathbb{Z},+,\cdot) \longrightarrow (\mathbb{Z}_n, +_n, \cdot_n)$ be homomorphisem s.t $f(a) = [a]$ fined Ker.f.

Soln :- $\text{Ker}.f = \{a \in \mathbb{Z} : f(a) = [0]\}$

$= \{a \in \mathbb{Z} : a \equiv 0 \pmod n\}$

$= \{a \in \mathbb{Z} : a = kn, k \in \mathbb{Z}\}$

$= \{0, \pm n, \pm 2n, \pm 3n, \dots$

Ex :- let $(R,+,\cdot)$ be aring with identaty ahomomor. $f_a : (R,+,\cdot) \longrightarrow (R,*,\cdot)$ $a \in R^*$ find Ker. f_a

Soln :- $\text{Ker}.f_a = \{x \in R : f_a(x) = a^*\}$

$= \{x \in R : a x a^{-1} = a^*\}$

$= \{x \in R : 1 \cdot x \cdot a^{-1} = 0 \cdot a\}$

$= \{x \in R : x \cdot 1 = 0\}$

$= \{x \in R : x = 0\}$

$= \{0\}$

Ex: let $f(Z_4, +_4, \cdot_4) \longrightarrow (Z_2, +_2, \cdot_2)$ be

ahomomorphisem defind by :-

$f(0) = f(2) = 0$

$f(1) = f(3) = 1$

find the Ker. f.

soln :- $Ker.f = \{x \in Z_4 : f(x) = 0\}$

$= \{x \in Z_4 : f(0) = f(2) = 0\}$.

$= \{0, 2\}$.