chapter Three

The homomorphisiem

Def :- Let $(R,+,\cdot)$ and $(\acute{R},\acute{+},\acute{\cdot})$ be two ring a function $f: R \longrightarrow \acute{R}$ is said to be a ring homomorphisiem iff :-

① $f(a+b) = f(a) \acute{+} f(b)$
② $f(a \cdot b) = f(a) \acute{\cdot} f(b)$
$\forall\, a,b \in R$

if f is homomorphisiem from R to $\acute{R}$ Then we say R is homomorphic to $\acute{R}$

EX :- Let $(Z,+,\cdot)$ and $(Z_n, +_n, \cdot_n)$ be two rings and $f: Z \longrightarrow Z_n$ defind by $f(a) = [a]$ Show That f is homo.

Soln :- Let $a,b \in Z$

1) $f(a+b) = [a+b] = [a] +_n [b]$

2) $f(a \cdot b) = [a \cdot b] = [a] \cdot_n [b]$

$\therefore f$ is homo

EX :- Let ~~the~~ $(\mathbb{Z}, +, \cdot)$ and $(\mathbb{Z}_e, +, \cdot)$ be two rings we define $f : \mathbb{Z} \longrightarrow \mathbb{Z}_e$ by $f(a) = 2a$.

Is f homo.

Soln :- Let $a, b \in \mathbb{Z}$

1) $f(a+b) = 2(a+b) = 2a + 2b = f(a) + f(b)$

2) $f(a \cdot b) = 2(a \cdot b) \neq (2a) \cdot (2b) = f(a) \cdot f(b)$

$\therefore$ f not homo.

EX :- Let $(R, +, \cdot)$ be any ring with identity for every invertible element $a \in R^*$, the function $f_a : R \longrightarrow R$ given by, $f_a(x) = ax \cdot a^{-1}$

Show that f_a is homomarphisiem.

Soln :- Let $x, y \in R$

1) $f_a(x+y) = a(x+y)a^{-1} = axa^{-1} + aya^{-1}$
$= f_a(x) + f_a(y)$

2) $f_a(x \cdot y) = a(x \cdot y)a^{-1} = ax \cdot (a^{-1}a) y a^{-1}$
$= (axa^{-1})(aya^{-1})$
$= f_a(x) \cdot f_a(y)$

$\therefore$ f is homo.

Ex:- let $(R,+,\cdot)$ be any ring with identity

The map $g: \mathbb{Z} \longrightarrow R$ where $(\mathbb{Z},+,\cdot)$ is a ring and g defined by $g(n) = n \cdot 1$. Show that g is homo.

Soln:- let $m, n \in \mathbb{Z}$

① $g(m+n) = (m+n)\cdot 1 = n\cdot 1 + m\cdot 1 = g(n)\cdot g(m)$

② $g(n\cdot m) = (n\cdot m)\cdot 1 = (m\cdot n)\cdot 1^2 = (n\cdot 1)(m\cdot 1)$

$= g(n)\cdot g(m)$

$\therefore g$ is homomorphisem.

Ex:- Show that the map $f: (\mathbb{Z}_4, +_4, \cdot_4) \longrightarrow (\mathbb{Z}_2, +_2, \cdot_2)$

defined by $f(0) = f(2) = 0$

$f(1) = f(3) = 1$

Soln:- $f(0+2) = f(2) = 0 = 0+0 = f(0) + f(2)$

$f(0+1) = f(1) = 1 = 0+1 = f(0) + f(1)$

$f(0+3) = f(3) = 1 = 0+1 = f(0) + f(3)$

$f(1+1) = f(2) = 0 = 0+0 = f(1) + f(1)$

Theorm(3·1):- Let f be a homomorphisiem from the ring $(R,+,\cdot)$ into the ring $(R',+',\cdot')$. Then

1) $f(0) = 0'$, where $0'$ is the Zero element of $(R',+',\cdot')$

2) $f(-a) = -f(a)$, $\forall$, $a \in R$

Proof:- $0 = 0 + 0$

$f(0) = f(0+0)$

$\Rightarrow f(0) = f(0) + f(0)$ [f is homo.]

$\Rightarrow f(0) - \underline{f(0)} = f(0) + f(0) - \underline{f(0)}$

$0' = f(0) + 0'$

$0' = f(0)$

② $a - a = 0$

$f(a-a) = f(0)$

$f(a) + f(-a) = 0'$ [f is homo. & by ①]

$\underline{-f(a) + f(a)} + f(-a) = 0' - f(a)$

$0' + f(-a) = -f(a)$

$f(-a) = -f(a)$

Theorm 3.3 :- If f is a homo- from a ring $(R,+,\cdot)$ onto a ring $(\acute{R},\acute{+},\acute{\cdot})$ if S is a subring of R Then $f(S)$ is a subring of $\acute{R}$.

proof :- $f(S) = \{f(b) \mid b \in S\}$

① Let $x, y \in f(S) \Rightarrow \exists\, a, b \in S$, s.t $x = f(a)$

, $y = f(b)$

$x - y = f(a) - f(b) = f(a - b)$

$\because$ S is subring in R and $a, b \in S \Rightarrow a - b \in S$

$\Rightarrow f(a-b) \in f(S) \Rightarrow x - y \in f(S)$

② $x \cdot y = f(a) \cdot f(b)$
$= f(a \cdot b)$

$\because$ S is subring in R , $a, b \in S \Rightarrow ab \in S$

$\Rightarrow f(a b) \in f(S)$

$\Rightarrow x - y \in f(S)$

$\therefore (f(S), +, \cdot)$ is subring of $\acute{R}$

Theorm (3.2) :- Let f be a homomorphism from the ring $(R, +, \cdot)$ with identity 1 onto the ring $(\acute{R}, \acute{+}, \acute{\cdot})$ with identity $\acute{1}$ Then :-

1) $f(1) = \acute{1}$

2) $f(a^{-1}) = f(a)^{-1}$, $\forall$ invertible element $a \in R$

Proof :- (1) $f(a) = \acute{1}\, f(a)$

$f(1 \cdot a) = \acute{1} \cdot f(a)$

$f(1) \cdot f(a) = \acute{1} \cdot f(a)$

by cancelation lows we get

$f(1) = \acute{1}$

(2) $a \cdot a^{-1} = 1$

$f(a \cdot a^{-1}) = f(1)$

$f(a) \cdot f(a^{-1}) = \acute{1}$ [f is homo - f by 1]

$f(a)^{-1} f(a) \cdot f(a^{-1}) = f(a)^{-1} \cdot \acute{1}$

$\acute{1} \cdot f(a^{-1}) = f(a)^{-1}$

$f(a^{-1}) = f(a)^{-1}$

Theorm 3.4 :- If $f:(R,+,\cdot) \longrightarrow (\acute{R},\acute{+},\acute{\cdot})$ is homo- and $\acute{S}$ is subring of $\acute{R}$ Then $f^{-1}(\acute{S})$ is subring of R.

proof :- let $a,b \in f^{-1}(\acute{S}) \Rightarrow f(a), f(b) \in \acute{S}$

$$f(a-b) = f(a) + f(-b) = f(a) - f(b)$$

$\because \acute{S}$ is subring of $\acute{R}$ Then $f(a)-f(b) \in \acute{S}$

$\therefore f(a-b) \in \acute{S} \Longrightarrow a-b \in f^{-1}(\acute{S})$

$\therefore f^{-1}(\acute{S})$ is a subring of R.