③ Every ideal of R is finitely generated.

Proof: $1 \Rightarrow 2$ suppose that R satisfies the A.c.c.
let C be a non-empty collection of ideals in R.
$\because C \neq \phi \Rightarrow \exists I_1 \in C \Rightarrow I_1$ is not maximal ideal.
$\Rightarrow \exists I_2 \in C$ s.t. $I_1 \subseteq I_2$ and I_2 is not maximal
$\Rightarrow \exists I_3 \in C$ s.t. $I_2 \subseteq I_3$ and I_3 is not maximal
and so on. Then we have $I_1 \subset I_2 \subset I_3 \subset \cdots \subset$!
Since R satisfies A.c.c.
$\therefore C$ has maximal element.

$2 \Rightarrow 3$ let I be an ideal of R and I is not finite generated.
$\because I \neq \phi \Rightarrow \exists a_1 \in I$ s.t. $I_1 = (a_1) \subset I$
$\Rightarrow \exists a_2 \in I$, $a_2 \notin I_1 \Rightarrow I_1 \subset I_2 = (a_1, a_2)$
Thus we get $I_1 \subset I_2 \subset I$
and so on we have $a_n \in I$, $a_n \notin I_{n-1}$ s.t.
$I_1 \subset I_2 \subset \cdots \subset I_{n-1} \subset I \subset \cdots$
let $C = \{I_n : n \in \mathbb{Z}\} \Rightarrow C \neq \phi$.
By ② C has maximal element
let I_K be a maximal element of C which is contradiction since $I_K \subset I_{K+1} \in C$
$\Rightarrow I$ must be finite generated.

$3 \Rightarrow 1$ Let $I_1 \subseteq I_2 \subseteq \cdots$ be the A.C.C. of ideals in R

Put $I = \bigcup_n I_n$, I is an ideal of R

By hypotheses I is finite generated

$\Rightarrow \exists\, a_1, a_2, \ldots, a_m \in R$ s.t.

$I = (a_1, a_2, \ldots, a_m)$

Assume that $a_1 \in I_{k_1}$, $a_2 \in I_{k_2}, \ldots, a_m \in I_{k_m}$

Let $r = \max\{k_1, \ldots, k_m\}$

$\Rightarrow a_1, a_2, \ldots, a_m \in I_r$

But $I_{r+1} \subseteq I$

$\Rightarrow I_{r+1} \subseteq I_r$ $[I \subseteq I_r]$

But $I_r \subseteq I_{r+1} \Rightarrow I_r = I_{r+1}$

$\therefore$ R satisfies the A.C.C.

Def: A ring $(R, +, \cdot)$ with identity is called Noetherian if R satisfies the A.C.C.

Ex: $(\mathbb{Z}, +, \cdot)$ is Noetherian

since $\mathbb{Z}$ is a ring with identity and

$(16) \subseteq (8) \subseteq (4) \subseteq (2)$

Any finite ring is Noetherian.

Proof: let $f(I_1) \subseteq f(I_2) \subseteq \cdots$ be A-c- in $f(R)$ where $I_1 \subseteq I_2 \subseteq \cdots$ be A-c- in R since R is Noetherian then $\exists\, k \in \mathbb{Z}_+$ s.t. $I_k = I_{k+1} = \cdots$

$\therefore f(I_k) = f(I_{k+1}) = \cdots$

$\because f(R)$ is a ring with 1.

$\therefore f(R)$ is Noetherian.

Def: let $(R,+,\cdot)$ be a ring and I be ~~a~~ a proper ideal of R then I is reducible if $\exists$ two ideals J and K in R s.t.

$$I = J \cap K \;,\; I \subset J \;,\; I \subset K$$

and I is irreducible if it is not reducible

Ex-: The ideal (6) is reducible in $\mathbb{Z}$.

$(6) = (2) \cap (3)$

$(6) \subset (2)$ and $(6) \subset (3)$

Ex-: Every prime ideal is irreducible.

Sol: Let P be a prime ideal and reducible

$\Rightarrow \exists\, I, J$ in R s.t. $P = I \cap J$, $P \subset I$, $P \subset J$

let $a \in I - P$ and $b \in J - P \Rightarrow a \cdot b \in I$, $a \cdot b \in J$,

$\Rightarrow$ it is a member of $P \Rightarrow P$ can not be a prime ideal c! since $a \in P$ or $b \in P$ (P is prime)

$\therefore$ every prime ideal is irreducible.