Differentiation (الاشتقاق)

قبل أن نبدأ بتعريف المشتقة نشير أننا تجنبنا كلياً مفهوم الغاية كما يعرف في كتب التفاضل والتكامل وأعطينا تعريف المشتقة مستخدمين مفهوم التقارب حيث إن هذا الاسلوب يظهر بوضوح تبسيط براهين مبرهنات مهمة مثل قاعدة السلسلة ومبرهنة الدالة النظيرة

Differentiability of the function (قابلية الدالة للاشتقاق)

Definition: Let $D \subset \mathbb{R}$ and $p \in D$

Let $F(x) = \dfrac{f(x) - f(p)}{x - p}$ (F is a function defined on $D \setminus \{p\}$)

If $\lim\limits_{x \to p} F(x)$ exists, then f is differentiable at p and

$$\lim_{x\to p} F(x) = \lim_{x\to p}\frac{f(x)-f(p)}{x-p} = f'(p)$$

$f'(p)$ is the derivative of f at p.
If f is differentiable at each point $x\in D$, then we say that f is differentiable on D.
$f'(x)$ is the derivative function of f on D.
$f'(p)$ is a real number & $f'(x)$ is a function.

Examples:

① $f(x) = c \quad \forall x\in\mathbb{R}$

$$\therefore f'(p) = \lim_{x\to p}\frac{f(x)-f(p)}{x-p} = \lim_{x\to p}\frac{c-c}{x-p} = 0 \Longrightarrow f'(x) = 0 \quad \forall x\in\mathbb{R}$$

② $f(x) = x, \quad \forall x\in\mathbb{R}$.

$$f'(p) = \lim_{x\to p}\frac{f(x)-f(p)}{x-p} = \lim_{x\to p}\frac{x-p}{x-p} = 1 \Longrightarrow f'(x) = 1 \quad \forall x\in\mathbb{R}$$

Examples: $f(x) = x^2 \quad \forall x\in\mathbb{R}$

$$f'(p) = \lim_{x\to p}\frac{f(x)-f(p)}{x-p} = \lim_{x\to p}\frac{x^2-p^2}{x-p} = \lim_{x\to p}(x+p) = 2p$$

$$\Longrightarrow f'(p) = 2p \Longrightarrow f'(x) = 2x \quad \forall x\in\mathbb{R}$$

Definition: Let $f: D\to\mathbb{R}$ be a function and D be an open interval. A function f is said to be differentiable at $x_0\in D$, if for any sequence $\langle x_n\rangle$ in D s.t. $x_n \to x_0$ ($x_n \neq x_0 \ \forall n$), then the sequence $\frac{f(x_n)-f(x_0)}{x_n-x_0}$ converges to the constant number $\alpha = \alpha(x_0)$.

$$\text{i.e. } \lim_{x_n\to x_0}\frac{f(x_n)-f(x_0)}{x_n-x_0} = \alpha(x_0) = \alpha = f'(x_0)$$

$$\text{or } \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \alpha(x_0) = f'(x_0)$$

Theorem: If f is differentiable at $x_0 \in D$, then f is continuous at x_0.

proof:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow \lim_{x \to x_0} (x - x_0) f'(x_0) = \lim_{x \to x_0} (f(x) - f(x_0))$$

$$\Rightarrow \lim_{x \to x_0} [f(x) - f(x_0)] = 0 \Rightarrow \lim_{x \to x_0} f(x) = f(x_0) \Rightarrow f \text{ is conts at } x_0$$

Remark: The converse of above theorem is not true. Consider the following example.

Example:

Let $f: \mathbb{R} \to \mathbb{R}$ be a function s.t. $f(x) = |x| \ \forall x \in \mathbb{R}$

$\Rightarrow f$ is conts on $\mathbb{R} \Rightarrow f$ is conts at $x_0 = 0$

But f is not differentiable at $x_0 = 0$, since

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x| - 0}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & x > 0 \\ \frac{-x}{x} = -1 & x < 0 \end{cases}$$

$\Rightarrow f'(0)$ does not exist $\Rightarrow f$ is not differentiable at $x_0 = 0$

Theorem: If $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ are two differentiable functions at $x_0 \in D$ and $c \in \mathbb{R}$, then:

① $f \mp g$ is differentiable at x_0 & $(f \mp g)'(x_0) = f'(x_0) \mp g'(x_0)$

② fg is differentiable at x_0 & $(fg)'(x_0) = f(x_0) g'(x_0) + g(x_0) f'(x_0)$

③ cf is differentiable at x_0 & $(cf)'(x_0) = c f'(x_0)$

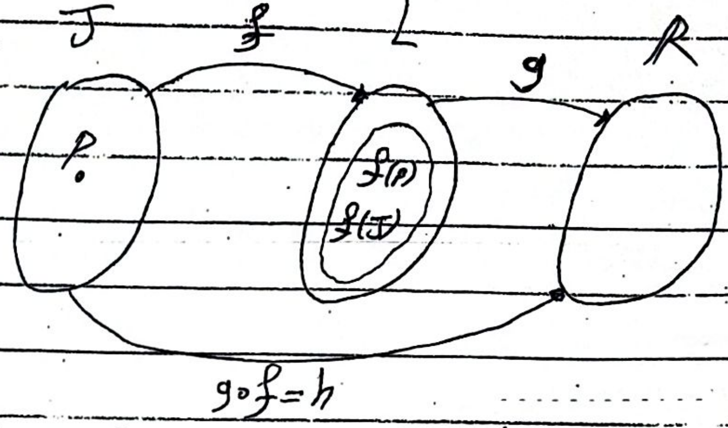
④ If $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0) f'(x_0) - f(x_0) g'(x_0)}{[g(x_0)]^2}$$

Also, $\frac{1}{g}$ is differentiable at x_0 & $\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{[g(x_0)]^2}$

Theorem (The chain rule) (مبرهنة السلسلة)

Let J and L be two open intervals in $\mathbb{R}$ and $f: J \to L$ and $g: L \to \mathbb{R}$ be two functions such that $f(J) \subset L$. If f is differentiable at $x_0 \in J$ and g is differentiable at $f(x_0) \in L$, then $g \circ f$ is differentiable at $x_0 \in J$ and $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$.



proof:

Let $h = g \circ f : J \to \mathbb{R}$, $x_0 \in J$

$$h'(x_0) = \lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0}$$

$$h'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \cdot \frac{f(x) - f(x_0)}{f(x) - f(x_0)}$$

$$= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= g'(f(x_0)) \cdot f'(x_0)$$

$$\Longrightarrow (g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Theorem (The inverse function theorem) (مبرهنة الدالة العكسية)

Let $f: D \to \mathbb{R}$ be a one-to-one function and differentiable at $x_0 \in D$ such that $f'(x_0) \neq 0$, then $f^{-1}: f(D) \to D$ is differentiable at $f(x_0)$ and $(f^{-1})'(f(x_0)) = \dfrac{1}{f'(x_0)}$ ch

$$(f^{-1})'(f(x_0)) = \lim_{x \to x_0} \frac{f^{-1}(f(x)) - f^{-1}(f(x_0))}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}}$$

$$= \frac{1}{f'(x_0)}$$

Ex $f(x)=x^2$ ...

Mean Value Theorems مبرهنات القيمة الوسطى

Theorem(1): Let $f: D \to \mathbb{R}$ be a differentiable function at $p \in D$. If $f'(p) > 0$, then there is $\epsilon > 0$ such that $f(x) < f(p)$ $\forall x < p$ in $N(p, \epsilon)$ and $f(x) > f(p)$ $\forall x > p$ in $N(x, p)$.

proof:

$x < p$: $f(x) < f(p)$; $x > p$: $f(x) > f(p)$, $f(p) < f(x)$

$\because f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} > 0 \Rightarrow \exists\, \epsilon > 0$ s.t

$\frac{f(x) - f(p)}{x - p} > 0 \quad \forall x \in N(p, \epsilon)$

$\Rightarrow f(x) - f(p)$ and $x - p$ have the same sign

If $x > p \Rightarrow f(x) > f(p)$

If $x < p \Rightarrow f(x) < f(p) \quad \forall x \in N(p, \epsilon)$

[i.e. f is increasing on $N(p, \epsilon)$]

Theorem(2): Let $f: D \to \mathbb{R}$ be a differentiable function at $p \in D$. If $f'(p) < 0$, then there is $\epsilon > 0$ such that $f(x) < f(p)$ $\forall x > p$ in $N(x, p)$ and $f(x) > f(p)$ $\forall x < p$ in $N(p, \epsilon)$.

proof:

$x < p$: $f(x) > f(p)$; $p < x$: $f(p) > f(x)$

$\because f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} < 0 \Rightarrow \exists\, \epsilon > 0$ s.t

$\frac{f(x) - f(p)}{x - p} < 0 \quad \forall x \in N(p, \epsilon)$

$\Rightarrow f(x) - f(p)$ and $x - p$ have the different sign

If $x > p \Rightarrow f(x) < f(p)$

If $x < p \Rightarrow f(x) > f(p) \quad \forall x \in N(p, \epsilon)$

[i.e. f is decreasing on $N(p, \epsilon)$]

Definition (A local maximum point)

A point p is called a local maximum point (l.m.p.) of f if there is a $N(p, \epsilon)$ such that $f(p) \geq f(x)$ $\forall x \in N(p, \epsilon)$.

② If $f(x)$ [illegible]

Definition (A local minimum point)

A point p is called a local minimum point (L.mi.p) of f, if there is a $N(p,\epsilon)$ such that $f(p) \leq f(x)\ \forall x \in N(p,\epsilon)$.

Examples: ①

Let $f: \mathbb{R} \to \mathbb{R}$ s.t $f(x) = 3x^4 - 4x^3 - 12x^2$

$f'(x) = 12x^3 - 12x^2 - 24x \Rightarrow f'(x) = 0 \Rightarrow 12x(x^2 - x - 2) = 0$

$\Rightarrow 12x(x+1)(x-2) = 0 \Rightarrow x = 0$ & $x = -1$ & $x = 2$

$f(0) \geq f(x)\ \forall x \in N(0,\epsilon) \Rightarrow 0$ is L.m.p. of f

$f(-1) \leq f(x)\ \forall x \in N(-1,\epsilon) \Rightarrow -1$ is L.mi.p. of f

$f(2) \leq f(x)\ \forall x \in N(2,\epsilon) \Rightarrow 2$ is L.mi.p. of f

min — max — min

$f'(x)$ $- - - - - -$ -1 $+ + +$ 0 $- - - - - - - -$ 2 $+ + + + +$

② Let $f: \mathbb{R}^+ \to \mathbb{R}$ s.t $f(x) = \sin(\frac{1}{x})$

$f'(x) = -\frac{1}{x^2}\cos(\frac{1}{x}) \Rightarrow f'(x) = 0 \Rightarrow \frac{-1}{x^2}\cos(\frac{1}{x}) = 0$

Sin

$\Rightarrow x = \frac{2}{(4n-1)\pi}$ is L.mi.p. of f $(n \in \mathbb{N})$

Sin

$\Rightarrow x = \frac{2}{(4n+1)\pi}$ is L.m.p. of f $(n \in \mathbb{N})$

Theorem: Let $f: D \to \mathbb{R}$ be a differentiable function at $p \in D$. If p is a local maximum point or a local minimum point, then $f'(p) = 0$

proof:

If $f'(p) > 0 \xrightarrow{Th} \exists$ a $N(p,\epsilon)$ s.t f is increasing function.

$\Rightarrow p$ is not a L.mi.p. and not a L.m.p. C!

If $f'(p) < 0 \xrightarrow{Th} \exists$ a $N(p,\epsilon)$ s.t f is decreasing function.

$\Rightarrow p$ is not a L.m.p. and not a L.mi.p. C!

$\Rightarrow f'(p) = 0$

Remarks: The converse of above theorem is not true. Consider the following examples.

Examples: Let $f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f'(x) = 0 \Rightarrow 3x^2 = 0$
$\Rightarrow x = 0 \Rightarrow f'(0) = 0$, but $x = 0$ is not a l.m.p. or a l.mi.p. of f

$+++ \quad +++$ (0 نقطة حرجة) $\quad f'(x) = 3x^2$

Rolle's theorem (مبرهنة رول)

Let f be a continuous function on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b) = 0$, then there is a point $c \in (a,b)$ such that $f'(c) = 0$.

proof:

Case (1): If f is constant $\Rightarrow f'(x) = 0 \ \forall x \in (a,b)$
$\Rightarrow \exists\, c \in (a,b)$ s.t $f'(c) = 0$.

Case (2): If f is not constant, then
$\because f$ is conts on a compact set $[a,b] \xrightarrow[\exists x_0, y_0 \in X \text{ s.t } f(y_0) < f(x) < f(x_0)\ \forall x \in X]{f: X \to R \text{ conts } \& X \text{ compact}} \exists\, x_0, y_0 \in [a,b]$ s.t x_0 is a l.m.p. and y_0 is a l.mi.p. s.t $f(y_0) < f(x) < f(x_0) \ \forall x \in [a,b]$

If $f(x_0) = f(y_0) \Rightarrow f$ is constant C!
$\Rightarrow f(x_0) \neq f(y_0) \Rightarrow x_0 \neq y_0$
$\Rightarrow$ at least one of the points x_0 or $y_0 \neq a$ or b ($f(a) = f(b)$)
$\Rightarrow$ either $x_0 \in (a,b)$ or $y_0 \in (a,b)$
$\xRightarrow{Th}$ either $f'(x_0) = 0$ or $f'(y_0) = 0$ (If p is l.m.p. or l.mi.p. $\Rightarrow f'(p) = 0$)
$\Rightarrow \exists\, c \in (a,b)$ s.t $f'(c) = 0$

Examples:

① $f(x) = 3x - x^3 \quad \forall x \in [-\sqrt{3}, \sqrt{3}]$
$\because f$ is conts on $[-\sqrt{3}, \sqrt{3}]$ & diff. on $(-\sqrt{3}, \sqrt{3})$ & $f(\sqrt{3}) = f(-\sqrt{3}) = 0$
$\therefore$ By Rolle's theorem $\exists\, c \in (-\sqrt{3}, \sqrt{3})$ s.t $f'(c) = 0$
$\because f'(x) = 3 - 3x^2 \Rightarrow f'(c) = 3 - 3c^2 \Rightarrow f'(c) = 0 \Rightarrow 3 - 3c^2 = 0 \Rightarrow$
$\Rightarrow c^2 = 1 \Rightarrow c = \mp 1$.

② $f(x) = \sqrt{1 - x^2} \quad \forall x \in [-1, 1]$ (H.w)

Lagrang's Mean Value Theorem

Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b), then there is a point $c \in (a,b)$ such that $f(b) - f(a) = (b-a) f'(c)$

proof:

Let $g : [a,b] \to \mathbb{R}$ be a function s.t

$g(x) = [f(b) - f(a)](x-a) - [f(x) - f(a)](b-a)$

$\therefore$ g is conts on $[a,b]$ and diff. on (a,b) and $g(a) = g(b) = 0$

$\therefore$ By Rolle's theorem $\exists\, c \in (a,b)$ s.t $g'(c) = 0$

$\Rightarrow g'(c) = [f(b) - f(a)] - f'(c)(b-a) = 0$

$\Rightarrow f(b) - f(a) = f'(c)(b-a)$

Example:

Let $f(x) = \sqrt{x} \quad \forall x \in [0,1]$

$\therefore$ f is conts on $[0,1]$ & diff. on $(0,1)$ $\Rightarrow \exists\, c \in (0,1)$ s.t

$f(b) - f(a) = (b-a) f'(c) \Rightarrow f(1) - f(0) = (1-0) f'(c) \Rightarrow 1 = \dfrac{1}{2\sqrt{c}}$

$\Rightarrow \sqrt{c} = \dfrac{1}{2} \Rightarrow c = \dfrac{1}{4} \in (0,1)$

$\therefore$ f is conts on $[x,b]$ & diff. on (x,b)

$\exists\, c \in (x,b)$ s.t $f(b) - f(x) = (b-x) f'(c)$

$\Rightarrow f(x) = f(b)$ constant

Corollary (1): If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f'(x) = 0 \ \forall x \in (a,b)$, then f is constant on $[a,b]$.

proof: Let $a \le x \le b$

$\therefore$ f is conts on $[a,x]$ and diff. on (a,x) $\xrightarrow{\text{Lagrange's}} \exists\, c \in (a,x)$ s.t

$f(x) - f(a) = (x-a) \underbrace{f'(c)}_{=0} = 0 \Rightarrow f(x) = f(a) \Rightarrow f$ is constant.

Corollary (2): If f is continuous on $[a,b]$ and differentiable on (a,b), then if $f'(x) > 0$ on (a,b), then f is increasing on (a,b) and if $f'(x) < 0$ on (a,b), then f is decreasing on (a,b).

proof: Let $x_1 < x_2$ in $[a,b] \Rightarrow f$ is conts on $[x_1, x_2]$ and diff. on (x_1, x_2)

$\xrightarrow{\text{Lagrange's}} \exists\, c \in (x_1, x_2)$ s.t $f(x_2) - f(x_1) = \underbrace{(x_2 - x_1)}_{>0} f'(c)$

If $f'(c) > 0 \Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow f(x_2) > f(x_1) \quad \forall x_2 > x_1$

$\Rightarrow f$ is increasing function on (a,b)

$f(x_2) - f(x_1)$ > 0 $f(x_2) - f(x_1)$

If $f'(c) < 0 \rightarrow f(x_2) - f(x_1) < 0 \rightarrow f(x_2) < f(x_1) \quad \forall x_2 > x_1$
$\rightarrow f$ is decreasing function on (a,b)

Taylor's Theorem   مبرهنة تايلر

If $f: D \rightarrow \mathbb{R}$ has $n+1$ continuous derivatives on D and $a, b \in D$, Then $\exists\, c$ between a and b such that:

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^n}{n!} f^{(n)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

If $b = x$, then

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

If $b = x$ & $a = 0 \rightarrow$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

Theorem (L'Hopital's Rule)   قاعدة لوبيتال

Let f, g be continuous functions on $[a,b]$ and differentiable on (a,b) and $f(a) = g(a) = 0$, if $g'(x) \neq 0 \ \forall x \in (a,b)$ and $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = L$, Then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = L$

Examples: $\lim\limits_{x \to 0} \frac{\sin 3x}{\sin 8x} = \lim\limits_{x \to 0} \frac{3 \cos 3x}{8 \cos 8x} = \frac{3}{8}$