Uniform continuity (8.1 الاستمرار المنتظم)

Definition: Let X be a metric space. A mapping $f: X \to \mathbb{R}$ is called uniformly continuous on X if for any $\epsilon > 0$, there is $\delta > 0$ (δ depends on ϵ) such that for each $x, y \in X$, if $d(x,y) < \delta$, then $|f(x) - f(y)| < \epsilon$.

Remark: Every uniformly continuous is continuous, but the converse is not true.

Example:

Let $f: \mathbb{R} \to \mathbb{R}$ be a mapping s.t $f(x) = x^2 \ \forall x \in \mathbb{R}$, then f is conts, but not uniformly conts.

proof: f is conts (ch:)

T.p f is not uniformly conts.

T.p $\exists \epsilon > 0, \forall \delta > 0$ s.t if $\exists x, y \in \mathbb{R}$ & $|x-y| < \delta \implies |f(x) - f(y)| \geq \epsilon$.

Let $\epsilon = 1$ & let $x = n$ & $y = n + \frac{1}{n}$ $\in \mathbb{R}$

$\therefore |x-y| = |n - (n + \frac{1}{n})| = \frac{1}{n} < \delta$ (by Archimedes).

But $|f(x) - f(y)| = |n^2 - (n + \frac{1}{n})^2| = |\cancel{n^2} - \cancel{n^2} - \frac{2n}{n} - \frac{1}{n^2}| = |-2 - \frac{1}{n^2}| = 2 + \frac{1}{n^2} > 1 = \epsilon$.

$\implies f$ is not uniformly conts.

يمكن جعل المثال السابق U.C. كالتالي:

Examples:

① If $f:(0,a] \to \mathbb{R}$ is a mapping s.t $f(x)=x^2 \ \forall x \in (0,a]$ & $a>0$

Then f is uniformly continuous.

proof T.p f is U.C

$\forall \epsilon>0, \exists \delta>0$ s.t $\forall x,y \in (0,a]$, if $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon$.

$\therefore |f(x)-f(y)| = |x^2-y^2| = |x-y||x+y|$

$$\le |x-y|(|x|+|y|)$$

$$< \delta(a+a) = 2a\delta$$

let $\epsilon = 2a\delta \Rightarrow \delta = \dfrac{\epsilon}{2a} \Rightarrow f$ is uniformly conts on $(0,a]$.

② If $f:(0,1) \to \mathbb{R}$ is a mapping s.t $f(x)=\dfrac{1}{x} \ \forall x \in (0,1)$

Then f is continuous, but f is not uniformly continuous.

proof f is conts to prove that f is not uniformly conts.

Let $\epsilon=1$ to prove that $\forall \delta>0, \exists x,y \in (0,1)$ s.t if $|x-y|<\delta \Rightarrow$

$|f(x)-f(y)| \geq \epsilon = 1$

$\therefore \delta>0 \xrightarrow{\text{A.P}} \exists n \in \mathbb{N}\ (n \geq 3)$ s.t $\dfrac{1}{n}<\delta$.

Let $\underset{\in(0,1)}{x}=\dfrac{1}{n}$ & $\underset{\in(0,1)}{y}=\dfrac{2}{n} \Rightarrow |x-y| = \left|\dfrac{1}{n}-\dfrac{2}{n}\right| = \dfrac{1}{n}<\delta$.

But $|f(x)-f(y)| = \left|n-\dfrac{n}{2}\right| = \dfrac{n}{2} > 1 \Rightarrow f$ is not U.C.

يمكن أن نجعل المثال الثاني U.C كالتالي:

③ If $f:[a,\infty) \to \mathbb{R}$ is a mapping s.t $f(x)=\dfrac{1}{x} \ \forall x \in [a,\infty)$, $a>0$

then f is uniformly continuous.

proof T.p f is U.C.

$\forall \epsilon>0, \exists \delta>0$ s.t $\forall x,y \in [a,\infty)$, if $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon$

$|f(x)-f(y)| = \left|\dfrac{1}{x}-\dfrac{1}{y}\right| = \left|\dfrac{y-x}{xy}\right| = \dfrac{|x-y|}{|xy|} < \dfrac{\delta}{a\cdot a} = \dfrac{\delta}{a^2}$

$a<x \Rightarrow \frac{1}{a}>\frac{1}{x}$

$a<y \Rightarrow \frac{1}{a}>\frac{1}{y}$

Let $\epsilon = \dfrac{\delta}{a^2} \Rightarrow \delta = \epsilon a^2 \Rightarrow f$ is U.C.

Theorem: Let X be a compact metric space. If $f:X \to \mathbb{R}$ is continuous on X, then f is uniformly continuous.

proof: Let X be a compact metric space & $f:X \to \mathbb{R}$ is conts.

Let $\epsilon>0$ to find $\delta>0$ s.t if $d(x,y)<\delta \Rightarrow |f(x)-f(y)|<\epsilon$

$\because \forall x \in X$, let $V = B(f(x), \frac{\epsilon}{2})$ be a ball in $\mathbb{R}$ $\xrightarrow{f \text{ conts}}$ $\exists$ a ball $N_x = B(x, \delta_x)$ in X s.t $f(N_x) \subseteq V$.

Let M_x be a ball in X s.t $M_x = B(x, \frac{\delta_x}{2})$.

$\Rightarrow X \subseteq \bigcup_{x \in X} M_x \Rightarrow \{M_x / x \in X\}$ is an open cover of X.

$\because X$ is compact $\Rightarrow \exists x_1, x_2, \dots, x_n \in X$ s.t $X \subseteq \bigcup_{i=1}^{n} M_{x_i}$.

Let $\delta = \min\{\frac{\delta_{x_i}}{2} / i = 1, 2, \dots, n\} \Rightarrow \delta > 0$.

Let $d(x,y) < \delta$.

$\because x \in X \subseteq \bigcup_{i=1}^{n} M_{x_i} \Rightarrow \exists k \ (1 \le k \le n)$ s.t $x \in M_{x_k}$.

$\Rightarrow d(x, x_k) < \frac{1}{2}\delta_{x_k}$.

$\delta \le \frac{1}{2}\delta_{x_k}$

$\delta + \frac{1}{2}\delta_{x_k} \le \frac{1}{2}\delta_{x_k} + \frac{1}{2}\delta_{x_k} = \delta_{x_k}$

$\because d(y, x_k) \le d(y, x) + d(x, x_k) < \delta + \frac{1}{2}\delta_{x_k} \le \delta_{x_k}$.

$\because f$ is conts $\Rightarrow |f(x) - f(x_k)| < \frac{\epsilon}{2}$ & $|f(y) - f(x_k)| < \frac{\epsilon}{2}$.

$\therefore |f(x) - f(y)| = |f(x) - f(x_k) + f(x_k) - f(y)| \le |f(x) - f(x_k)| + |f(x_k) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

$\Rightarrow f$ is uniformly conts.

Corollary: if $f : [a,b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.

Intermediate value property (خاصية القيمة الوسطى)

Let $f : [a,b] \to \mathbb{R}$ be a real continuous mapping on $[a,b]$, then for each real numbers s between $f(a)$ and $f(b)$, there is $z \in [a,b]$ such that $f(z) = s$.

proof:

Suppose that $f(a) < s < f(b)$. By the same way if $f(b) < s < f(a)$

devid $I_1 = [a,b]$ into two closed intervals which are equal in the length $[a,m]$ and $[m,b]$ $\left(m = \frac{a+b}{2}\right)$.

if $f(m) = s \Rightarrow \exists z = m$ s.t $f(m) = f(z) = s$.

And if $f(m) \ne s \Rightarrow f(m) < s$ or $f(m) > s$

$\therefore$ we get closed interval $I_2 = [a_2, b_2]$ s.t

$I_2 = [a_2, b_2] = [a, m]$ if $f(a) < s < f(m)$.

$I_2 = [a_2, b_2] = [m, b]$ if $f(m) < s < f(b)$.

$\Rightarrow f(a_2) < s < f(b_2)$

By mathematical indiction, we get a sequence $\langle I_n \rangle$ of closed

$\& \; g(a) = f(a) - a > 0 \;\&\; g(b) = f(b) - b < 0 \xrightarrow{\text{المتوسطة}} \exists x_0 \in (\ldots$ $\ldots = x_0$ ...t Theorem

intervals s.t $I_n \subseteq I_{n-1}$ & $|I_n| = \frac{b-a}{2^{n-1}}$ & $f(a_n) < s < f(b_n)$

$\because |I_n| \to 0$, then by nested intervals $\bigcap I_n = \{z\}$

$\Rightarrow z \in I_1 = [a,b]$.

T.p $f(z) = s$.

$\because \langle |I_n| \rangle \to 0 \Rightarrow a_n \to z$ & $b_n \to z$.

$\because f$ is conts $\Rightarrow f(a_n) \to f(z)$ & $f(b_n) \to f(z)$.

لأن $[a_n,b_n]$ أصغر فترة تقترب من نقطة التقاطع لأن متتابعة الأطوال تقترب من الصفر

But $f(a_n) < s \;\forall n \Rightarrow f(z) \leq s$.

Also, $s < f(b_n) \;\forall n \Rightarrow s \leq f(z)$

$\Rightarrow f(z) = s$.

بعض تطبيقات مبرهنة القيمة المتوسطة

① Let $f: \mathbb{R} \to \mathbb{R}$ be conts s.t $f(-x) = -f(x) \;\forall x \in \mathbb{R}$, then $\exists x_0 \in \mathbb{R}$ s.t $f(x_0) = 0$.

الدالة فردية

Solu If $f(y) > 0 \Rightarrow f(-y) < 0 \Rightarrow f(-y) < 0 < f(y) \xrightarrow[\text{المتوسطة}]{\text{مبرهنة القيمة}} \exists x_0 \in [-y, y]$ s.t $f(x_0) = 0$.

② Let $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$ (n odd).

Then $p(x) = 0$ has at least one real root.

Solu

هذه النتيجة غير صحيحة اذا كانت

$\exists x_1 \in \mathbb{R}$ s.t $p(x_1) > 0$ & $\exists x_2 \in \mathbb{R}$ s.t $p(x_2) < 0$.

درجة المعادلة زوجية.

$\Rightarrow p(x_2) < 0 < p(x_1) \xrightarrow[\text{المتوسطة}]{\text{مبرهنة القيمة}} \exists x_0 \in \mathbb{R}$ s.t $p(x_0) = 0$.

المعادلة $x^2+1=0$ ليس لها جذر حقيقي

③ فوق