Sequences And Series of Functions

متتابعات ، متسلسلات الدوال

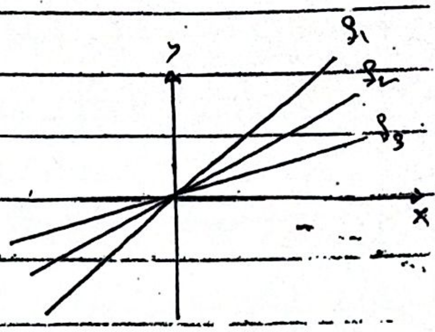
Let $S \subseteq \mathbb{R}$ and $X = \{f \mid f : S \to \mathbb{R}$ be a function$\} = F(S)$ be the set of all real functions defined on S.

$\therefore \forall x \in S$, $\langle f_n(x) \rangle$ is a sequence of functions in X.

Pointwise Convergence التقارب النقطي

Definition: A sequence of functions $\langle f_n(x) \rangle$ is said to be pointwise converges (or converges) to $f(x)$, if for each $\epsilon > 0$ and each $x \in S$, there is $K \in \mathbb{N}$ (K depends on x and ϵ) such that $|f_n(x) - f(x)| < \epsilon \ \forall n > K$.

or $\lim\limits_{n \to \infty} f_n(x) = f(x) \ \forall x \in S$ or $f_n(x) \longrightarrow f(x)$



Examples: ① Let $f_n(x) = \dfrac{x}{n} \ \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$.

Then $f_n(x)$ converges to $f(x) = 0$.

Sol: Let $\epsilon > 0$ & $x \in \mathbb{R}$ to find $K \in \mathbb{N}$ s.t $|f_n(x) - f(x)| < \epsilon \ \forall n > K$.

$\therefore |f_n(x) - f(x)| = \left|\dfrac{x}{n} - 0\right| = \left|\dfrac{x}{n}\right| = \dfrac{|x|}{n} < \dfrac{|x|}{K} \quad \forall n > K$

$K\epsilon > |x| \Rightarrow \epsilon > \frac{|x|}{K} \Rightarrow \frac{|x|}{K} < \epsilon$

But $\dfrac{|x|}{K} < \epsilon$ (by Archimedes) $\Longrightarrow |f_n(x) - f(x)| < \epsilon \ \forall n > K$.

$\therefore K > \dfrac{|x|}{\epsilon} \Longrightarrow f_n(x) \longrightarrow 0$

② Let $f_n(x) = \dfrac{nx}{1 + n^2 x^2} \ \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$.

Then $f_n(x) \longrightarrow f(x) = 0$

Sol:

If $x = 0 \Longrightarrow f_n(0) = 0 \longrightarrow 0$

If $x \neq 0$ & let $\epsilon > 0$ to find $K \in \mathbb{N}$ s.t $|f_n(x) - f(x)| < \epsilon \ \forall n > K$.

$\therefore |f_n(x) - f(x)| = \left|\dfrac{nx}{1 + n^2x^2} - 0\right| = \left|\dfrac{nx}{1 + n^2x^2}\right| < \left|\dfrac{nx}{n^2x^2}\right| = \left|\dfrac{1}{nx}\right| = \dfrac{1}{n|x|} < \dfrac{1}{K|x|} \quad \forall n > K$

$K\epsilon > \frac{1}{|x|} \Rightarrow \epsilon > \frac{1}{K|x|} \Rightarrow \frac{1}{K|x|} < \epsilon$

But $\dfrac{1}{K|x|} < \epsilon$ (by Archimedes) $\Longrightarrow |f_n(x) - f(x)| < \epsilon \ \forall n > K$.

$\therefore K > \dfrac{1}{|x|\epsilon} \Longrightarrow f_n(x) \longrightarrow 0$

Uniform Convergence   التقارب المنتظم

Definition: A sequence of functions $\langle f_n(x) \rangle$ is called uniformly converges to $f(x)$, if for each $\epsilon > 0$, there is $K \in \mathbb{N}$ (K depends on ϵ) such that $|f_n(x) - f(x)| < \epsilon \ \forall x \in S$ & $\forall n > K$ and we write $f_n \xrightarrow{u.c} f$.

Examples: ① Let $f_n(x) = \frac{x}{n}$ $\forall x \in (0,b)$ & $\forall n \in N$.

Then $f_n(x) \xrightarrow{u.c} f(x) = 0$

Solu

Let $\epsilon > 0$ to find $K \in N$ s.t $|f_n(x) - f(x)| < \epsilon$ $\forall n > K$, $\forall x \in (0,b)$

$\therefore |f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{|x|}{n} < \frac{b}{n} < \frac{b}{K}$ $\forall n > K$.

$K\epsilon > b \Rightarrow \epsilon > \frac{b}{K} \Rightarrow \frac{b}{K} < \epsilon$

But $\frac{b}{K} < \epsilon$ (by Archimedes) $\Rightarrow |f_n(x) - f(x)| < \epsilon$ $\forall n > K$, $\forall x \in (0,b)$

$\therefore K > \frac{b}{\epsilon} \Rightarrow f_n(x) \xrightarrow{u.c} 0$

② Let $f_n(x) = \frac{nx}{1+n^2x^2}$ $\forall x \in [a,\infty)$, $a > 0$.

Then $f_n(x) \xrightarrow{u.c} f(x) = 0$

Solu

Let $\epsilon > 0$ to find $K \in N$ s.t $|f_n(x) - f(x)| < \epsilon$ $\forall n > K$, $\forall x \in [a,\infty)$

$a < x$
$\frac{1}{a} > \frac{1}{x}$

$\therefore |f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \left|\frac{nx}{1+n^2x^2}\right| < \left|\frac{nx}{n^2x^2}\right| = \frac{1}{nx} < \frac{1}{na} < \frac{1}{Ka}$ $\forall n > K$.

$K\epsilon > \frac{1}{a} \Rightarrow \epsilon > \frac{1}{aK} \Rightarrow \frac{1}{aK} < \epsilon$

But $\frac{1}{Ka} < \epsilon$ (by Archimedes) $\Rightarrow |f_n(x) - f(x)| < \epsilon$ $\forall n > K$, $\forall x \in [a,\infty)$

$\therefore K > \frac{1}{a\epsilon} \Rightarrow f_n(x) \xrightarrow{u.c} 0$

Remarks: Every uniformly convergent sequence is pointwise convergent but the converse is not true.

Examples:

Let $f_n(x) = \frac{nx}{1+n^2x^2}$ $\forall x \in R$, $\forall n \in N$

$\Rightarrow f_n(x)$ is pointwise converges to $f(x) = 0$, but is not uniformly converges to $f(x) = 0$, since

if $x_n = \frac{1}{n}$ $\forall n \in N$ & $\epsilon = \frac{1}{4}$ $\Rightarrow \left|f_n(\frac{1}{n}) - f(\frac{1}{n})\right| = \left|\frac{1}{2} - 0\right| = \frac{1}{2} \not< \frac{1}{4}$

مبرهنات التقارب المنتظم

Theorem (1): Let $\langle f_n \rangle$ be a sequence of real bounded functions on S, if $\langle f_n \rangle$ is uniformly converges to f, then f is bounded.

proof: T.p f is bounded

$\because \langle f_n \rangle$ is U.C $\Rightarrow \exists K \in \mathbb{N}$ s.t $|f_n(x) - f(x)| < 1 \ \forall x \in S, \forall n \geq K$.

$\because f_n$ is bounded $\forall n \in \mathbb{N} \Rightarrow \exists M > 0$ s.t $|f_{K+1}(x)| \leq M \ \forall x \in S$.

$\therefore |f(x)| = |f(x) - f_{K+1}(x) + f_{K+1}(x)| \ \forall x \in S$.

$$\leq |f(x) - f_{K+1}(x)| + |f_{K+1}(x)| \leq 1 + M$$

$\Rightarrow |f(x)| \leq 1 + M \ \forall x \in S \Rightarrow f$ is bounded function on S.

Theorem (2): Let $\langle f_n \rangle$ be a sequence of real continuous functions on S, if $\langle f_n \rangle$ is uniformly converges to f, then f is also continuous on S.

proof: T.p f is conts on S.

i.e. T.p if $\langle x_m \rangle$ is a sequence in S s.t $x_m \to x_0$, then $f(x_m) \to f(x_0)$.

Let $\epsilon > 0$, since $\langle f_n \rangle$ is U.C $\Rightarrow \exists n_0 \in \mathbb{N}$ s.t $|f_n(x) - f(x)| < \frac{\epsilon}{3} \ \forall x \in S, \forall n \geq n_0$

$\because f_n$ is conts $\forall n \in \mathbb{N}$ & $x_m \to x_0$ in $S \Rightarrow f_{n_0+1}(x_m) \to f_{n_0+1}(x_0)$ in $\mathbb{R}$

i.e. $\exists K \in \mathbb{N}$ s.t $|f_{n_0+1}(x_m) - f_{n_0+1}(x_0)| < \frac{\epsilon}{3} \ \forall m \geq K$

$\therefore |f(x_m) - f(x_0)| = |f(x_m) - f_{n_0+1}(x_m) + f_{n_0+1}(x_m) - f_{n_0+1}(x_0) + f_{n_0+1}(x_0) - f(x_0)|$

$$\leq |f(x_m) - f_{n_0+1}(x_m)| + |f_{n_0+1}(x_m) - f_{n_0+1}(x_0)| + |f_{n_0+1}(x_0) - f(x_0)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \forall m \geq K$$

$\Rightarrow f(x_m) \to f(x_0) \Rightarrow f$ is conts on S

نعطي الآن شروطاً كافية لكي يكون التقارب تقارباً منتظماً

Theorem: Let $\langle f_n \rangle$ be a sequence of real continuous functions on S, if $\langle f_n \rangle$ converges to f (f be continuous function on S), S is a compact subset of $\mathbb{R}$ and $f_{n+1}(x) \geq f_n(x)$ $\forall x \in S, \forall n \in \mathbb{N}$ or $f_{n+1}(x) \leq f_n(x)$ $\forall x \in S, \forall n \in \mathbb{N}$, then $\langle f_n \rangle$ is uniformly converges to f

proof: To prove the theorem when $f_{n+1}(x) \leq f_n(x)$ $\forall x \in X, \forall n \in \mathbb{N}$.

Let $g_n(x) = f_n(x) - f(x)$, $\forall x \in X, \forall n \in \mathbb{N}$.

$\Rightarrow g_n(x)$ is conts & $\langle g_n(x) \rangle \to 0 = g(x)$.

Also $g_{n+1}(x) \leq g_n(x)$ $\forall x \in X, \forall n \in \mathbb{N}$.

It is enough to prove that $\langle g_n \rangle$ is uniformly converges to $g(x) = 0$.

Let $\epsilon > 0$ $\xRightarrow{g_n(x) \text{ conv.}}$ $\forall x \in S, \exists n(x) \in \mathbb{N}$ s.t $|g_{n(x)}(x)| < \frac{\epsilon}{2}$.

$\because g_n(x)$ is conts on $S \Rightarrow g_{n(x)}(x)$ is conts at each $x \in S$.

$\Rightarrow \exists$ open interval J_x of x s.t if $y \in J_x \Rightarrow |g_{n(x)}(y) - g_{n(x)}(x)| < \frac{\epsilon}{2}$

$\therefore |g_{n(x)}(y)| = |g_{n(x)}(y) - g_{n(x)}(x) + g_{n(x)}(x)| \leq |g_{n(x)}(y) - g_{n(x)}(x)| + |g_{n(x)}(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Let $F = \{J_x \mid x \in S\} \Rightarrow F$ is an open cover of $S \Rightarrow S \subseteq \bigcup_{x \in S} J_x$

$\xRightarrow{S \text{ compact}}$ $\exists \{J_{x_i} \mid i = 1, 2, \ldots, m\}$ is a finite subcover of S.

$\Rightarrow \exists\, n(x_1), n(x_2), \ldots, n(x_m) \in \mathbb{N}$

Let $K = \max\{n(x_1), n(x_2), \ldots, n(x_m)\}$

$\Rightarrow |g_K(y)| < \epsilon$ $n>K$

$\because g_{n+1}(x) \leq g_n(x)$ $\forall x \in S, \forall n \in \mathbb{N}$.

$\Rightarrow |g_n(y)| \leq |g_K(y)| < \epsilon$ $\forall n > K$.

$\Rightarrow |g_n(y)| < \epsilon$ $\forall n > K, \forall y \in S \Rightarrow \langle g_n \rangle$ is uniformly converges to $g(y) = 0 \Rightarrow \langle f_n \rangle$ is uniformly converges to f on S.