Continuous Real Valued Mappings

التطبيقات الحقيقية المستمرة - الفضاء $C(X)$

Let X be a metric space, then $C(X) = \{f \mid f : X \to \mathbb{R} \text{ is conts}\}$ is the set of all continuous real valued mappings.

$C(X) \neq \phi$, since $\exists\, f \in C(X)$ s.t $f(x) = c \;\forall x \in X$ is conts.

Theorem: If f and g are continuous real valued mappings, then:

① $f+g$ is continuous s.t $(f+g)(x) = f(x) + g(x)$.

② $f.g$ is continuous s.t $(f.g)(x) = f(x).g(x)$.

③ $\forall a \in \mathbb{R}$, af is continuous s.t $(af)(x) = a f(x)$.

④ If $g(x) \neq 0$, then $\frac{f}{g}$ is continuous s.t $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

⑤ $|f|$ is continuous s.t $|f|(x) = |f(x)|$.

proof:

① T.p $f+g : X \to \mathbb{R}$ is conts

Let $x_0 \in X$ and $\langle x_n \rangle$ be a sequence in X s.t $x_n \to x_0$.

$\because f$ is conts $\Longrightarrow f(x_n) \to f(x_0)$ in $\mathbb{R}$.

$\because g$ is conts $\Longrightarrow g(x_n) \to g(x_0)$ in $\mathbb{R}$.

$\xRightarrow{\text{By th}} f(x_n) + g(x_n) \to f(x_0) + g(x_0)$ in $\mathbb{R}$.

$\Longrightarrow (f+g)(x_n) \to (f+g)(x_0) \Longrightarrow f+g$ is conts at x_0.

Remarks: ① From ① & ② in above theorem $(C(X), +, .)$ is a vector space.

② Any polynomial $p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ is conts.

Real Mappings on compact spaces

التطبيقات الحقيقية المعرفة على فضاءات مرصوصة

Definition (Bounded mapping)

A mapping $f : X \to \mathbb{R}$ is called bounded if there is $M > 0$ such that $|f(x)| \leq M \;\forall x \in X$.

or R_f is bounded in $\mathbb{R}$.

Examples: A mapping $f:(0,1) \to \mathbb{R}$ s.t $f(x) = 2x \ \forall x \in (0,1)$ is bounded, since $\exists\, 2 > 0$ s.t $|f(x)| = |2x| \le 2 \ \forall x \in (0,1)$.

Theorem: Let X and X' be metric spaces and $f: X \to X'$ be a continuous mapping, if X is compact, then $f(X)$ is compact.

proof Let X be a compact space.

T.P $f(X)$ is compact.

Let $\{V_\alpha \mid \alpha \in \Lambda\}$ be any open cover of $f(X)$.

$\Rightarrow f(X) \subseteq \bigcup\limits_{\alpha \in \Lambda} V_\alpha$ & $V_\alpha \underset{\text{open}}{\subseteq} X' \ \forall \alpha \in \Lambda$.

$\Rightarrow f^{-1}(f(X)) \subseteq f^{-1}(\bigcup\limits_{\alpha \in \Lambda} V_\alpha) \Rightarrow X \subseteq \bigcup\limits_{\alpha \in \Lambda} f^{-1}(V_\alpha)$.

$\because f$ is conts & $V_\alpha \underset{\text{open}}{\subseteq} X' \ \forall \alpha \in \Lambda \Rightarrow f^{-1}(V_\alpha) \underset{\text{open}}{\subseteq} X \ \forall \alpha \in \Lambda$.

$\Rightarrow \{f^{-1}(V_\alpha) \mid \alpha \in \Lambda\}$ is an open cover of X.

$\because X$ is compact $\Rightarrow \exists \{f^{-1}(V_{\alpha_i}) \mid i = 1, 2, \dots, n\}$ is a finite subcover.

i.e. $X \subseteq \bigcup\limits_{i=1}^{n} f^{-1}(V_{\alpha_i}) \Rightarrow f(X) \subseteq \bigcup\limits_{i=1}^{n} V_{\alpha_i} \Rightarrow f(X)$ is compact.

Theorem: A mapping $f: X \to \mathbb{R}$ is bounded if f is continuous on a compact space X.

proof: $\because f: X \to \mathbb{R}$ is conts and X is compact, then by above theorem $f(X)$ is compact in $\mathbb{R} \Rightarrow f(X)$ is bounded (any compact subset of a metric space is bounded)

$\Rightarrow \exists M > 0$ s.t $|f(x)| \le M \ \forall x \in X \Rightarrow f$ is bounded.

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Remark: If $f: X \to \mathbb{R}$ is continuous and X is not compact, then f is not necessary bounded. Consider the following examples.

① $f:(0,1) \to \mathbb{R}$ s.t $f(x) = \frac{1}{x} \ \forall x \in (0,\infty) \Rightarrow f$ is conts and not bounded, since $\forall M > 0, \exists n \in \mathbb{N}$ s.t $f(\frac{1}{n}) = n > M$ $(n \ge 1)$ or $f(X) = f((0,\infty)) = (0,\infty)$ not bounded f is not bounded

② $f(x) = 2x \ \forall x \in (0,1) \Rightarrow f$ is conts and bounded, since $\exists\, 2 > 0$ s.t $|2x| \le 2 \ \forall x \in (0,1)$

or $f(X) = f((0,1)) = (0,2)$ is bounded $\Rightarrow f$ is bounded

Theorem: If $f: X \to \mathbb{R}$ is a real continuous mapping and X is a compact space, then there are $x_0, y_0 \in X$ such that $f(y_0) \leq f(x) \leq f(x_0) \quad \forall x \in X$

y_0 نقطة نهاية صغرى للتطبيق f — $f(y_0)$ قيمة النهاية الصغرى للتطبيق f

x_0 نقطة نهاية عظمى للتطبيق f — $f(x_0)$ قيمة النهاية العظمى للتطبيق f

Proof:

$\because f$ is cont's and X is compact $\xrightarrow{th} f(X) = Y$ is compact in $\mathbb{R}$

$\Rightarrow f(X) = Y$ is bounded and closed [any compact subset of a metric space is closed and bounded]

$\Rightarrow Y$ has sup and inf.

Let $\sup Y = M \Rightarrow M \in Y$ (Y closed).

$\Rightarrow \exists x_0 \in X$ s.t $f(x_0) = M$.

Also, let $m = \inf Y \Rightarrow m \in Y$ (Y closed).

$\Rightarrow \exists y_0 \in X$ s.t $f(y_0) = m$.

$\Rightarrow \exists x_0, y_0 \in X$ s.t $f(y_0) \leq f(x) \leq f(x_0) \quad \forall x \in X$.