Riemann Integral تكامل ريمان

Def: (1) A partition of the interval $[a,b]$ is a set
$p = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ s.t
$x_0 < x_1 < x_2 < \dots < x_n$

(2) $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ are called segments of p فترات التجزئة

(3) $\Delta x_i = |x_{i-1} - x_i| \quad \forall i = 1, 2, \dots, n$
is called the length of the segment $[x_{i-1}, x_i]$

(4) $\Delta p = \max\{\Delta x_i / i = 1, 2, \dots, n\}$ is called the norm of p

(5) If $p^* = \{x_0, x_1, x_2, y, x_3, z, x_4, \dots, x_n\} \Rightarrow p \subset p^*$

Def: A partition p^* is called a refinement of p if $p \subset p^*$ تنعيم للتجزئة
and $\Delta p^* \leq \Delta p$.

Now, let f be a bounded function on $[a,b]$ and let
$m = \inf\{f(x) / x \in [a,b]\}$
$M = \sup\{f(x) / x \in [a,b]\}$
Since f is bounded function on $[a,b]$, then f is bounded on each $[x_{i-1}, x_i]$ and let
$m_i = \inf\{f(x) / x \in [x_{i-1}, x_i]\}$
$M_i = \sup\{f(x) / x \in [x_{i-1}, x_i]\}$

$\Rightarrow m \leq m_i \leq M_i \leq M \quad \forall i = 1, 2, \dots, n$

let $\underline{R}(f, p) = \sum_{i=1}^{n} m_i \Delta x_i$ = lower Riemann sum of f on $[a,b]$ with a partition p (المجموعة الريماني الأسفل للدالة f على الفترة $[a,b]$ بالنسبة للتجزئة p)

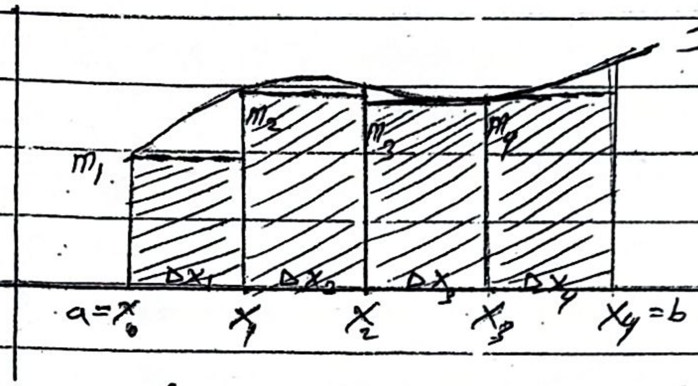
Let $\bar{R}(f,p)=\sum_{i=1}^{n} M_i \Delta x_i$ = upper Riemann sum of f on $[a,b]$ with a partition p (المجموع الريماني الأعلى للدالة f على الفترة $[a,b]$ بالنسبة للتجزئة p)

Remarks:

① $\underline{R}(f,p) \leq \bar{R}(f,p)$ (since $m_i \leq M_i \ \forall i$)

② $m(b-a) \leq \underline{R}(f,p) \leq \bar{R}(f,p) \leq M(b-a)$

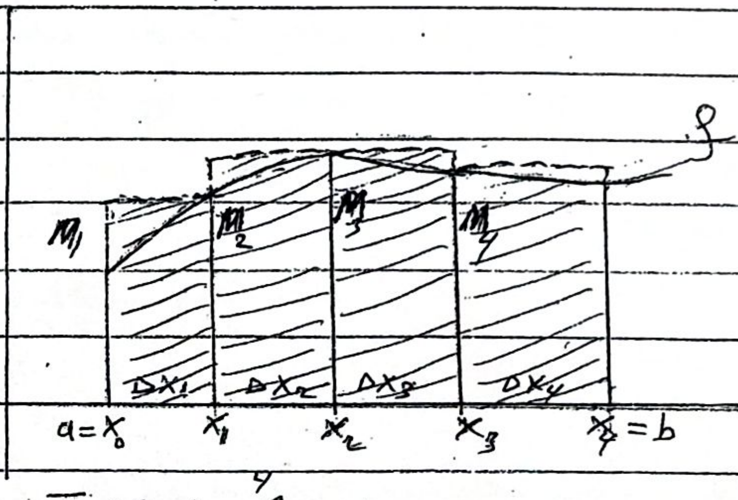
مجموع ريمان الأسفل هو عبارة عن مجموع مساحة المستطيلات الواقعة تحت منحنى الدالة f كما في الشكل التالي:



$\underline{R}(f,p)=\sum_{i=1}^{4} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 + m_4 \Delta x_4$

$p=\{a=x_0, x_1, x_2, x_3, x_4=b\}$

مجموع ريمان الأعلى هو عبارة عن مجموع مساحة المستطيلات الواقعة فوق منحنى الدالة f كما في الشكل التالي:



$\bar{R}(f,p)=\sum_{i=1}^{4} M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 + M_4 \Delta x_4$

Remarks:

① $\because \underline{R}(f,P) \leq M(b-a) \Rightarrow \underline{R}(f,P)$ is bounded above by $M(b-a)$
$\Rightarrow \underline{R}(f,P)$ has sup i.e.

$\sup\{\underline{R}(f,P) \,/\, P$ is a partition on $[a,b]\} = R\underline{\int} f = \underline{\int_a^b} f$ = lower Riemann integral of f on $[a,b]$.

It is clear that $\underline{R}(f,P) \leq \underline{\int_a^b} f$

② $\because m(b-a) \leq \overline{R}(f,P) \Rightarrow \overline{R}(f,P)$ is bounded below by $m(b-a)$
$\Rightarrow \overline{R}(f,P)$ has inf i.e.

$\inf\{\overline{R}(f,P) \,/\, P$ is a partition on $[a,b]\} = R\overline{\int} f = \overline{\int_a^b} f$ = upper Riemann integral of f on $[a,b]$

It is clear that $\overline{\int_a^b} f \leq \overline{R}(f,P)$

③ $m(b-a) \leq \underline{R}(f,P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \overline{R}(f,P) \leq M(b-a)$

Definition: Let $f: [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$. If $\underline{\int_a^b} f = \overline{\int_a^b} f$, then f is said to be Riemann integrable (or R integrable) on $[a,b]$ and is denoted by $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$

Theorem: If f is a bounded function on $[a,b]$ and P^* is a refinement of P, then $\underline{R}(f,P) \leq \underline{R}(f,P^*)$ and $\overline{R}(f,P^*) \leq \overline{R}(f,P)$

proof: let $p = \{a = x_0, x_1, x_2, \ldots, x_{u-1}, x_u, \ldots, x_{n-1}, x_n = b\}$ be a partition on $[a, b]$

let $p^* = \{x_0, x_1, x_2, \ldots, x_{u-1}, \bar{x}, x_u, \ldots, x_{n-1}, x_n\}$ be a refinement of p where $x_{u-1} < \bar{x} < x_u$

To prove that $\underline{R}(f, p) \leq \underline{R}(f, p^*)$

$\underline{R}(f,p) = \sum_{i=1}^{n} m_i \Delta x_i$ where m_i is $\inf f$ on $[x_{i-1}, x_i]$

let $m' = \inf f$ on $[x_{u-1}, \bar{x}]$ & $m'' = \inf f$ on $[\bar{x}, x_u]$

& $m_u = \inf f$ on $[x_{u-1}, x_u] \longrightarrow m_u \leq m'$ & $m_u \leq m''$

$$\underline{R}(f, p^*) = \sum_{i=1}^{u-1} m_i \Delta x_i + (\bar{x} - x_{u-1}) m' + (x_u - \bar{x}) m'' + \sum_{i=u+1}^{n} m_i \Delta x_i$$

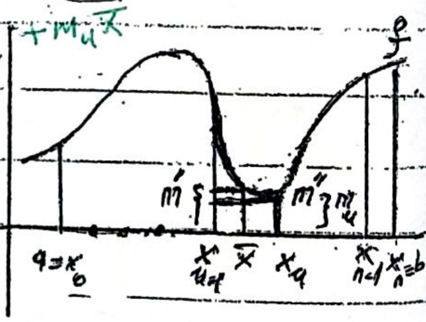
بإضافة وطرح حدة الشكل $m_u \underbrace{(x_u - x_{u-1})}_{\Delta x_u}$ ينتج

$$\underline{R}(f, p^*) = \left(\sum_{i=1}^{u-1} m_i \Delta x_i + m_u (x_u - x_{u-1}) + \sum_{i=u+1}^{n} m_i \Delta x_i \right) + (\bar{x} - x_{u-1}) m' + (x_u - \bar{x}) m'' - m_u \underbrace{(x_u - x_{u-1})}_{\Delta x_u}$$

$\bar{x}m' - x_{u-1}m' + x_u m'' - \bar{x}m'' - m_u x_u + m_u x_{u-1}$ $+ m_u \bar{x}$

$$= \sum_{i=1}^{n} m_i \Delta x_i + (m' - m_u)(\bar{x} - x_{u-1}) + (m'' - m_u)(x_u - \bar{x})$$

$m'\bar{x} - m_u\bar{x} - m'x_{u-1} + m_u x_{u-1} + m'' x_u - m_u x_u - m''\bar{x} + \bar{x} m_u$



$\therefore \underline{R}(f, p^*) = \underline{R}(f, p) + (m' - m_u)(\bar{x} - x_{u-1}) + (m'' - m_u)(x_u - \bar{x})$

$\therefore \underline{R}(f, p^*) - \underline{R}(f, p) = \underbrace{(m' - m_u)}_{\geq 0} \underbrace{(\bar{x} - x_{u-1})}_{> 0} + \underbrace{(m'' - m_u)}_{\geq 0} \underbrace{(x_u - \bar{x})}_{> 0}$

$\Rightarrow \underline{R}(f, p^*) - \underline{R}(f, p) \geq 0 \Longrightarrow \underline{R}(f, p) \leq \underline{R}(f, p^*)$

To prove that $\overline{R}(f,P^*) \leq \overline{R}(f,P)$

$\overline{R}(f,P) = \sum_{i=1}^{n} M_i \Delta x_i$ where M_i is sup f on $[x_{i-1}, x_i]$

Let $M' = \sup f$ on $[x_{u-1}, \bar{x}]$ & $M'' = \sup f$ on $[\bar{x}, x_u]$

& $M_u = \sup f$ on $[x_{u-1}, x_u] \Rightarrow M' \leq M_u$ & $M'' \leq M_u$

$$\overline{R}(f,P^*) = \sum_{i=1}^{u-1} M_i \Delta x_i + M'(\bar{x} - x_{u-1}) + M''(x_u - \bar{x}) + \sum_{i=u+1}^{n} M_i \Delta x_i$$

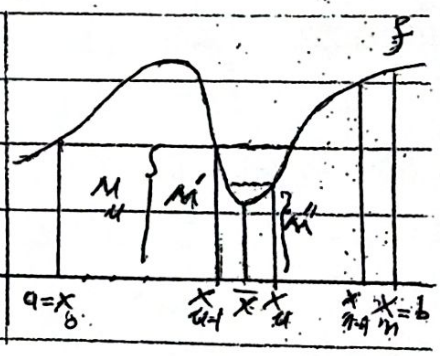
$$= \left(\sum_{i=1}^{u-1} M_i \Delta x_i + M_u(x_u - x_{u-1}) + \sum_{i=u+1}^{n} M_i \Delta x_i\right) + M'(\bar{x} - x_{u-1})$$

$$+ M''(x_u - \bar{x}) - M_u(x_u - x_{u-1})$$

$$= \overline{R}(f,P) + (M' - M_u)(\bar{x} - x_{u-1}) + (M'' - M_u)(x_u - \bar{x})$$

$$\overline{R}(f,P^*) - \overline{R}(f,P) = \underbrace{(M' - M_u)}_{\leq 0}\underbrace{(\bar{x} - x_{u-1})}_{>0} + \underbrace{(M'' - M_u)}_{\leq 0}\underbrace{(x_u - \bar{x})}_{>0}$$

$$\Rightarrow \overline{R}(f,P^*) - \overline{R}(f,P) \leq 0 \Rightarrow \overline{R}(f,P^*) \leq \overline{R}(f,P)$$



<u>Corollary</u>: If f is a bounded function on $[a,b]$, then $\underline{R}(f,P_1) \leq \overline{R}(f,P_2)$ for any partitions P_1 and P_2 of $[a,b]$.

<u>Proof</u>:

Let $P = P_1 \cup P_2 \Rightarrow P$ is a refinement of P_1 & P_2.

$\underline{R}(f,P_1) \leq \underline{R}(f,P)$ (By theorem (1)) ⊕