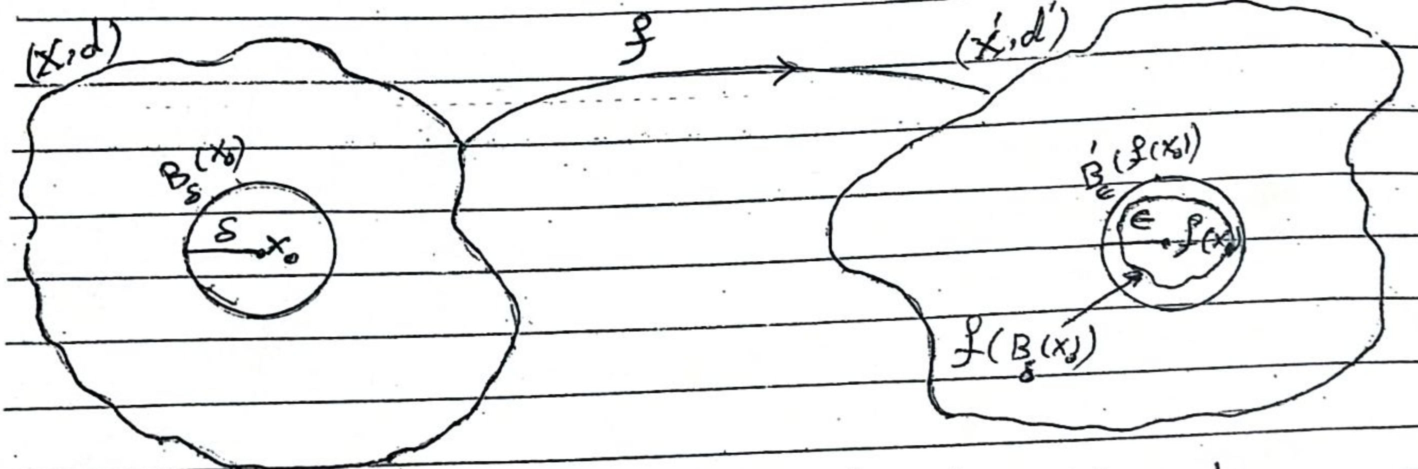
Continuity محاضرة 8

Definition: Let (X,d) and (X',d') be metric spaces. A mapping $f: X \to X'$ is called continuous at a point $x_0 \in X$, if for any $\epsilon > 0$, there is $\delta > 0$ (δ depends on ϵ and x_0) such that for each $x \in X$, if $d(x,x_0) < \delta$, then $d'(f(x), f(x_0)) < \epsilon$.

or

$f: X \to X'$ is continuous at $x_0 \in X \Leftrightarrow$ for any ball $B'_\epsilon(f(x_0))$ with center $f(x_0)$ and radius ϵ in X', there is a ball $B_\delta(x_0)$ with center x_0 and radius δ in X such that $f(B_\delta(x_0)) \subseteq B'_\epsilon(f(x_0))$



Remark: A mapping $f: X \to X'$ is called continuous (or continuous on X) iff f is continuous at each point in X.

Theorem: A mapping $f: (X,d) \to (X',d')$ is continuous iff for each open set U in X', $f^{-1}(U)$ is open set in X.

i.e.

$f: X \to X'$ is conts $\Leftrightarrow \forall U \underset{open}{\subseteq} X'$, then $f^{-1}(U) \underset{open}{\subseteq} X$.

where $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$

proof $\longrightarrow$

Suppose that f is conts on X & $U \underset{open}{\subseteq} X'$.

T.P $f^{-1}(U) \underset{open}{\subseteq} X$.

let $x_0 \in f^{-1}(U) \to f(x_0) \in U \xrightarrow[\text{open}]{U \text{ is}} \exists\, \epsilon > 0$ s.t $B'_\epsilon(f(x_0)) \subset U$.

∵ f is conts $\to \exists$ a ball $B_\delta(x_0)$ in X s.t $f(B_\delta(x_0)) \subseteq B'_\epsilon(f(x_0)) \subset U$.

$\to f(B_\delta(x_0)) \subset U \to B_\delta(x_0) \subseteq f^{-1}(U) \to f^{-1}(U) \underset{\text{open}}{\subseteq} X$.

$\Leftarrow$ suppose that $\forall\, U \underset{\text{open}}{\subseteq} X'$, then $f^{-1}(U) \underset{\text{open}}{\subseteq} X$.

T.P f is conts.

let $x_0 \in X$ & $B'_\epsilon(f(x_0))$ be a ball in $X' \to B'_\epsilon(f(x_0))$ is an open set in X'

$\xrightarrow{\text{hyp}} f^{-1}(B'_\epsilon(f(x_0)))$ is an open set in X and $x_0 \in f^{-1}(B'_\epsilon(f(x_0)))$.

$\to \exists\, \delta > 0$ s.t $B_\delta(x_0) \subset f^{-1}(B'_\epsilon(f(x_0))) \to f(B_\delta(x_0)) \subset B'_\epsilon(f(x_0))$.

$\to f$ is conts.

Theorem: A mapping $f: (X, d) \to (X', d')$ is continuous iff for each closed set F in X', then $f^{-1}(F)$ is closed set in X.

proof: $\Rightarrow$

suppose that f is conts and F is closed set in X'.

T.P $f^{-1}(F)$ is closed in X.

∵ F is closed in $X' \to X' - F$ is open in $X' \xrightarrow{\text{above th}} f^{-1}(X' - F)$ is open in X. But $f^{-1}(X' - F) = X - f^{-1}(F) \to f^{-1}(F)$ is a closed set in X.

$\Leftarrow$ T.P f is conts.

let U be an open set in $X' \to X' - U$ is closed in $X' \xrightarrow{\text{hyp}} f^{-1}(X' - U)$ is closed in X. But $f^{-1}(X' - U) = X - f^{-1}(U) \to f^{-1}(U)$ is open in X.

$\xrightarrow{\text{above th}} f$ is conts.

Convergence and continuity (التقارب و الاستمرارية)

Theorem: A mapping $f: (X, d) \to (X', d')$ is continuous at $x_0 \in X$ iff for every sequence $\langle x_n \rangle$ in X converges to x_0, then the sequence $\langle f(x_n) \rangle$ in X' converges to $f(x_0)$.

proof: $\Rightarrow$

suppose that f is conts at x_0 and $\langle x_n \rangle$ is a sequence in X s.t

$x_n \to x_0$, to prove that $f(x_n) \to f(x_0)$ in X'.
Let U be an open set in X' s.t $f(x_0) \in U \xRightarrow{f \text{ is conts}} f^{-1}(U)$ is open in X
and $x_0 \in f^{-1}(U)$. Since $x_n \to x_0 \Rightarrow f^{-1}(U)$ contains all but a finite
number of the terms of $\langle x_n \rangle \Rightarrow U$ contains all but a finite number
of the terms of $\langle f(x_n) \rangle \Rightarrow f(x_n) \to f(x_0)$.

$\Longleftarrow$

Suppose that if $x_n \to x_0$ in X, then $f(x_n) \to f(x_0)$ in X'.
T.p f is conts at x_0.
Suppose that f is not conts at $x_0 \Rightarrow \exists\, \epsilon > 0$ s.t $\forall n \in N$
$f(B_{\frac{1}{n}}(x_0)) \not\subseteq B'_{\epsilon}(f(x_0))$
i.e. $\forall n \in N$, $\exists\, x_n \in X$ s.t if $d(x_n, x_0) < \frac{1}{n}$, then
$d'(f(x_n), f(x_0)) \geq \epsilon$
$\Rightarrow f(x_n) \not\to f(x_0)$, but $x_n \to x_0$ C!
[Since let $\epsilon > 0 \xRightarrow{\text{Archi}} \exists\, k \in N$ s.t $\frac{1}{k} < \epsilon \Rightarrow d(x_n, x_0) < \frac{1}{n} < \frac{1}{k} < \epsilon \ \forall n > k$].
∴ f is conts at x_0.

Theorem: Let $f : (X, d) \to (X', d')$ and $g : (X', d') \to (X'', d'')$ be mappings such that f is continuous at $x_0 \in X$ and g is continuous at $f(x_0) \in X'$, then $g \circ f$ is continuous at $x_0 \in X$.

proof: T.p $g \circ f : (X, d) \to (X'', d'')$ is conts at $x_0 \in X$.
Let $\langle x_n \rangle$ be a sequence in X s.t $x_n \to x_0$.
∴ f is conts at $x_0 \xRightarrow{\text{above th}} f(x_n) \to f(x_0)$.
∴ g is conts at $f(x_0) \xRightarrow{\text{above th}} g(f(x_n)) \to g(f(x_0))$.
$\Rightarrow (g \circ f)(x_n) \to (g \circ f)(x_0) \xRightarrow{\text{above th}} g \circ f$ is conts at x_0.

Examples: ① Let $f : (X, d) \to (X', d')$ be the constant mapping s.t $f(x) = c \ \forall x \in X$, then f is continuous.

Sol^n ①:
$\forall \epsilon > 0$, we can find $\delta > 0$ s.t if $d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \epsilon$
∴ $d'(f(x), f(x_0)) = d'(c, c) = 0 < \epsilon$

$\therefore$ We can choose δ any positive number

(2) iiik

T.p $f: X \to X'$ is conts

Let $U \underset{open}{\subseteq} X' \Rightarrow f^{-1}(U) = \begin{cases} \phi & \text{if } c \notin U \\ X & \text{if } c \in U \end{cases}$

$\therefore \phi$ & X are open sets in $X \Rightarrow f^{-1}(U)$ is open in $X \Rightarrow f$ is conts

(2) Let $f: (X,d) \to (X,d)$ be the identity mapping s.t $f(x) = x \ \forall x \in X$

Then f is continuous

Sol. (1) iiik

$\forall \epsilon > 0, \exists \delta > 0$ s.t if $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \epsilon$

$\therefore d(f(x), f(x_0)) = d(x, x_0) < \delta = \epsilon$

$\therefore$ We can choose $\delta = \epsilon$

(2) iiik

T.p $f: X \to X$ is conts.

Let $V \underset{open}{\subseteq} X \Rightarrow f^{-1}(V) = \{x \in X / f(x) \in V\} = \{x \in X / x \in V\} = V \underset{open}{\subseteq} X$

$\Rightarrow f$ is conts

(3) Let $f: \mathbb{R} \to \mathbb{R}$ be a mapping s.t $f(x) = x^2 \ \forall x \in \mathbb{R}$, then f is conts.

proof: Let $x_0 \in \mathbb{R}$.

$\therefore \forall \epsilon > 0, \exists \delta > 0$ s.t if $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| = |x^2 - x_0^2|$

$= |x - x_0||x + x_0|$

Let $\delta = 1 \Rightarrow |x - x_0| < 1 \Rightarrow |x| < |x_0| + 1$

$\therefore |x + x_0| \leq |x| + |x_0| < 2|x_0| + 1$

$\therefore |f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0||x + x_0| < \delta(2|x_0| + 1)$.

Let $\delta(2|x_0| + 1) = \epsilon \Rightarrow \delta = \dfrac{\epsilon}{2|x_0| + 1}$

$\therefore$ We can choose $\delta = \min\{1, \dfrac{\epsilon}{2|x_0| + 1}\}$.

(4) Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a mapping s.t $f(x) = \dfrac{1}{x} \ \forall x \in \mathbb{R}^+$, then f is continuous at $x = 2$.

proof: $\forall \epsilon > 0, \exists \delta > 0$ s.t if $|x - 2| < \delta \Rightarrow |f(x) - f(2)| < \epsilon$

$\therefore |f(x) - f(2)| = \left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|2-x|}{2|x|}$

Let $\delta = 1 \Rightarrow |x-2| < 1$.

$\therefore |2| = |2 - x + x| \leq |2-x| + |x| < 1 + |x| \Rightarrow |x| > 1$

$\therefore \frac{|2-x|}{2|x|} < \frac{|2-x|}{2} < \frac{\delta}{2}$

Let $\epsilon = \frac{\delta}{2} \Rightarrow \delta = 2\epsilon \Rightarrow \delta = \min\{1, 2\epsilon\}$

(5) Let $f: \mathbb{R} \to \mathbb{R}$ be a mapping such that:-

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Is f continuous at $x = 0$

Solu

f is not conts at $x = 0$, since $\frac{1}{n} \to 0$, but $f(\frac{1}{n}) = 1 \to 1 \neq f(0) = 0$

(6) Let $f: [a,b] \to \mathbb{R}$ be a mapping such that

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then f is not continuous.

proof Let $x_0 \in [a,b]$

If $x_0 \notin \mathbb{Q} \Rightarrow \exists\, x_n \in \mathbb{Q}$ s.t $x_n \to x_0$, but $f(x_n) = 1 \to 1 \neq f(x_0) = 2$

Also, if $x_0 \in \mathbb{Q} \Rightarrow \exists\, x_n \in \mathbb{Q}^c$ s.t $x_n \to x_0$, but $f(x_n) = 2 \to 2 \neq f(x_0) = 1$

$\Rightarrow f$ is not conts