Proposition (4.16):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$, then S is closed iff S contains all its cluster points (i.e $\bar{S} = S$)

Proof: $\Rightarrow$) suppose the result is not true i.e $\exists$ a cluster point p for S such that $p \notin S$, $(p \in X - S)$.

$\because$ S is closed, then $X - S$ is open, hence $(X - S) \cap S = \emptyset$ C! (p is a cluster point for S).

$\Leftarrow$) let $l(S) \subseteq S$, T.P S is closed i.e $X - S$ is open.

Let $x \in X - S$, $x \notin S$ $i.e$ $x \notin l(S)$, x is not a cluster point.

$\exists$ open set U_x ; $x \in U_x$ and $U_x \cap S = \emptyset$, $\therefore$ $U_x \subseteq X - S$.

In particular $\exists$ a ball $B(x) \subseteq X - S$ $\rightarrow$ $X - S$ is open

$\therefore$ S is closed.

Example:

$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\right\}$ is not closed.

$X - S$ is not open, $\exists$ $0 \neq x \in X - S$, $\exists$ any ball $B(x) \nsubseteq X - S$

$\therefore$ $0 \notin X - S$

Definition (4.17):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$ and $p \in X$, define.

$$d(p, S) = \inf\{d(p, s) : s \in S\}$$

is called the distant between the point p and the set S

Remark (4.18):

If $S \subseteq X$, (X,d) be a metric space and $p \in S$, then $d(p,S) = 0$

$$d(p,S) = \inf \{d(p,s): s \in S\}$$

If $p \in S$, then $\inf \{d(p,p): \} = \inf\{0: positive\ number\} = o$.

The converse of remark (4.18) is not true in general as the following example show:

Example:

Let $S = (a,b)$, $X = R$.

$d(a,S) = \inf \{d(a,s): a < s < b\}$

$= \inf \{p - p + \epsilon, p - p + 2\epsilon, \cdots\}$

$= \inf \{\epsilon, \cdots\} = 0$

Proposition (4.19):

Let (X,d) be a metric space and $\emptyset \neq S \subseteq X, p \in X$, then $d(S,p) = 0$ iff $p \in S$ or p is a cluster point of S.

Proof: ⇒) $d(S,p) = 0$ suppose that $p \notin S$ T.P p is a cluster point for S.

If p is not a cluster point for S.

∃ a ball $B_r(p)$ such that $B_r(p) \cap S = \emptyset$

$\therefore d(s,p) > r \quad , \quad s \in S \qquad C!$ (since $d(S,p) = 0$)

$\therefore p$ is a cluster point for S.

$\Leftarrow$) If $p \in S$ by remark (4.18) $d(S,p) = 0$.

If p is a cluster point for S, then for any open set U, $p \in U$

$(U - \{p\} \cap)S \neq \emptyset$

In particular $\exists$ a ball $B_\epsilon(p)$; $B_\epsilon(p) \cap S \neq \emptyset$

$\exists s \neq p \in S \quad , \quad s \in B_\epsilon(p)\,,\ d(s,p) < \epsilon$

$d(S,p) = \inf \{d(s,p) < \epsilon, +, +, \cdots\}$

$\qquad = 0$

Corollary (4.20):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$, then

$\bar{S} = \{x \in X: d(S,x) = 0\}$ $d(S,p) = 0$.

Proof: $\bar{S} = S \cup l(S)$ by proposition (4.19) ($d(S,x) = 0$ iff $x \in X$ or x is a cluster point for S).