Let (X, d) be a metric space and $S \subseteq X$, then S is open iff S is a union of balls

Proof:- $\Rightarrow$)let S be an open set

Then $\forall x \in S$, $\exists\, r_x > 0$ such that $B_{r_x}(x) \subseteq S$

$\therefore \bigcup_{x \in S} B_{r_x}(x) = S$.

$\Leftarrow$) $S = \bigcup_{i \in w} B_i$ are balls

$\because$ every ball is an open set $\Rightarrow S = \bigcup_{i \in w} B_i$ is open (by proposition (4.8)).

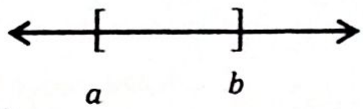
Definition (4.13):-

Let (X, d) be a metric space (topological space) and $E \subseteq X$, then E is closed in X if $X - E$ is open in X.

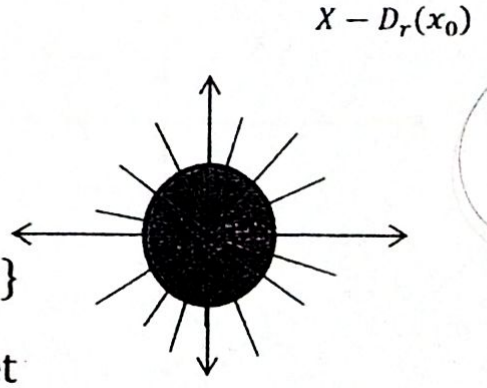
Examples:-

1- $[a, b] \subset R$, $[a, b]$ is closed

Since $R - [a, b] = (-\infty, a) \cup (b, \infty)$ is open.



The union of open set in a metric space is open.



In general any disk is a closed set.

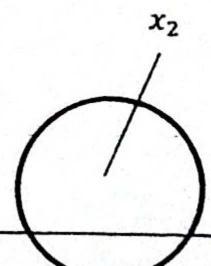
$$D_r(x_0) = \{x \in X: d(x, x_0) \leq r\}$$

$X - D_r(x_0) = \{x \in X: d(x, x_0) > r\}$ is an open set

2- Every finite subset E of a metric space (X, d) is a closed set.

Proof:- let $E = \{x_1, x_2, \cdots, x_n\} \subseteq X$

T.P $X - E$ is open.



Let $a \in X - E$, $\therefore a \neq x_i$, $\forall i = 1,2,\cdots,n$

$\therefore \exists\, 0 < d_i$, $\forall i = 1,2,\cdots,n$

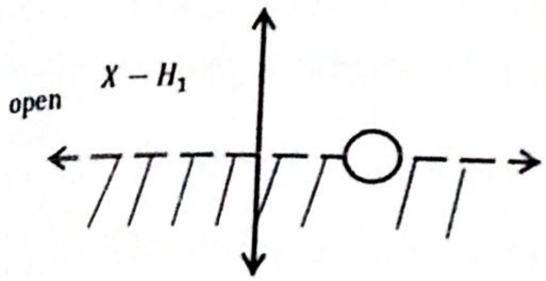
Take $r = \min\{d_1, d_2, \cdots, d_n\} \Rightarrow B_r(a) \nsubseteq E$

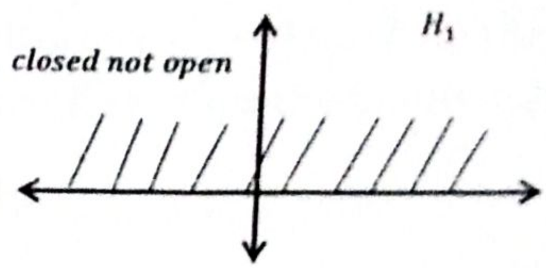
$\Rightarrow B_r(a) \cap E = \emptyset \Rightarrow B_r(a) \subseteq X - E \Rightarrow X - E$ is open

$\Rightarrow E$ is closed.

3- $H_1 = \{ (x,y) \in R^2 : x \in R \ , \ y \geq 0\}$

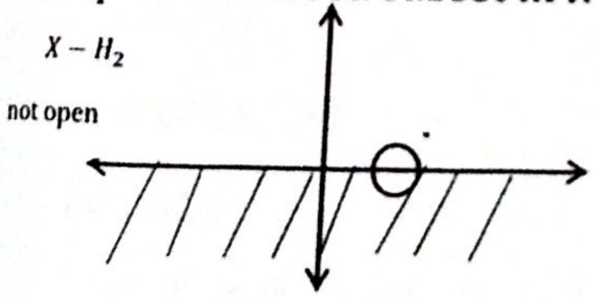
is closed not open subset in R^2.

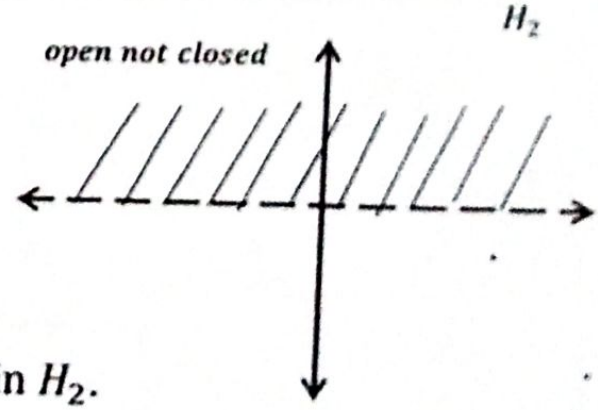




Since the ball with center $(x, 0)$ is not contain in H_1.

$H_2 = \{ (x,y) \in R^2 : x \in R \ , \ y > 0\}$
is open not closed subset in R^2





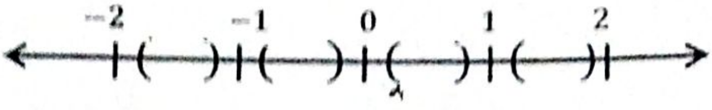
Since the ball with center (x, y) is contain in H_2.

4- $Q \subset R$ is not closed $R - Q = Q'$ is not open

$\therefore Q$ is not closed

5- Z (Integers number) is closed

$R - Z = \cdots \cup (-1,0) \cup (0,1) \cup (1,2) \cup \cdots$



Balls $\Rightarrow$ open

$\therefore Z$ is closed

6- $X, \emptyset$ are closed sets

$X - X = \emptyset$.is open $\therefore X$ is closed, $X - \emptyset = X$ is open, $\therefore \emptyset$ is closed.

Proposition (4.14):-

Let (X, d) be a metric space (topological space) and let T be the collection of all closed subsets of X. Then T satisfies the followings:

1) $X, \emptyset \in T$ (i.e, X and $\emptyset$ are closed)

2) The union of finite numbers of elements in T is an element is T (i.e, the union of finite numbers of closed set is again a closed set)

3) The intersection of finite or infinite numbers of elements of T is an element in T

(i.e, the intersection of finite or infinite numbers of closed set is closed)

Proof: (H.w)

Remark:

Let $X \neq \emptyset$ *and* $y_\alpha \subseteq X$ $\forall \alpha \in \Lambda$ then

$X - \cup_{\alpha \in \Lambda} y_\alpha = \cap_\alpha (X - y_\alpha)$

$X - \cap_{\alpha \in \Lambda} y_\alpha = \cup_\alpha (X - y_\alpha)$.

Definition (4.15):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$ and $p \in X$, we say.