$(\sum_n a_n + \sum_n b_n) + (-\sum_n a_n) = \sum_n b_n$ convergence series by (3.3) C!.

$\therefore \sum_n a_n + \sum_n b_n$ is a divergence series.

H.W: Give an example of two divergence series such that the sum of them is convergence.

Some Kinds of Test:

1-The Convergence Test:

Proposition (3.5):

If $\sum_n a_n$ is a convergence series, then the sequence $< a_n >$ is converge to zero.

Proof: let $< S_n >$ be sequence of partial sum of $\sum_n a_n$

$\because \sum_n a_n$ is a convergence series, then $< S_n >$ is a convergence sequence and hence $< S_n >$ is a Cauchy sequence.

$i.e \ \forall \in > 0 \ \exists k \in Z^+ \ k = k(\in)$ s.t $|S_n - S_m| < \in \ \forall n, m > k$.

In particular put $m = n + 1$

$|S_n - S_m| = |-a_{n+1}| = |a_{n+1}| = |a_{n+1} - 0| < \in \ \forall n > k$

$< a_n >$ converge to zero.

* If $<a_n> \not\to 0 \Rightarrow \sum_n a_n$ is diverge

Example:

$<\frac{n}{2n+1}>$ is converge to $\frac{1}{2}$

$\therefore \sum_{n=1}^{\infty} \frac{n}{2n+1}$ is a diverges series, since not converge to zero.

Remark:

In general the converse of proposition (3.5) is not true.

Example: $<\frac{1}{n}> \to 0$

$<\frac{1}{n}>$ is converge to 0, but $\sum_{n=1}^{\infty}\frac{1}{n}$ is a diverges series since harmonic.

2-The Comparison Test:

Proposition (3.7):

Let $\sum_n a_n$ and $\sum_n b_n$ be series, if:

1. $0 \leq a_n \leq b_n \;\forall n$ and $\sum_n b_n$ is a convergence series, then $\sum_n a_n$ is a convergence series.
2. $0 \leq b_n \leq a_n \;\forall n$ and $\sum_n b_n$ is a divergence series, then $\sum_n a_n$ is a divergence series.

Proof: **1.** let $< S_n >$ and $< T_n >$ be sequences of partial sum of $\sum_n a_n$ and $\sum_n b_n$ respectively.

$\because \sum_n b_n$ is a convergence series, then $< T_n >$ is a convergence sequences that converge to T.

$0 \leq a_n \leq b_n \;\forall n$.

$S_n = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i = T_n \to T$

$S_n \leq T \quad \forall n$, then $< S_n >$ is bounded

$S_1 = a_1$ $\qquad S_1 \leq S_2$

$S_2 = a_1 + a_2$ $\qquad S_2 \leq S_3$

$S_3 = a_1 + a_2 + a_3$ $\qquad S_1 \leq S_2 \leq \cdots \leq S_n$

$\vdots$

$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$

$\therefore < S_n >$ is non-decreasing (increasing) sequence, hence $< S_n >$ is a convergence sequence.

$\therefore \sum_n a_n$ is a convergence series.

Proof: **2.** Suppose the result is not true, $i.e$ $\sum_n a_n$ is a convergence series.

By part (1) $\sum_n b_n$ is a convergence series C!

Example:

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p > 0$, $p \in R$, this series is called a series of type-p. If $p=1 \Rightarrow \sum \frac{1}{n^p}$ is divergs series.

When $p = 2$

$\sum_{n=1}^{\infty} \frac{1}{n^2} \geq \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$

$(n+1)^2 = n^2 + 2n + 1 \geq n^2 + n = n(n+1)$

$\therefore \frac{1}{(n+1)^2} < \frac{1}{n(n+1)} \quad \forall n \in N$

$\because \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is a converging series, then the comparison test

$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a converging series.

When $0 < p < 1$

$n^p < n \quad \forall n \in N \ , \ 0 < \frac{1}{n} < \frac{1}{n^p} \quad \forall n$

$\because \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergence series (harmonic)

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ is a divergence series by (comparison test).

When $p > 2$

$n^2 < n^p \quad n \in N \ , \ 0 < \frac{1}{n^p} < \frac{1}{n^2}$

$\because \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergence series

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ is a convergence series by (comparison test).

3-The Absolute and Conditional Convergence Test:

Definition (3.8):

1) A series $\sum_n a_n$ is said to be absolutely convergence if $\sum_n |a_n|$ is a convergence series.

2) If $\sum_n a_n$ is a convergence series and $\sum_n |a_n|$ is a divergence series, then we say that $\sum_n a_n$ is a conditionally convergence series.

Examples:

1. $\sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^{n-1}$ (Geometric series). Is it absolutely convergence?

$\sum_{n=1}^{\infty}\left|\left(\frac{-1}{2}\right)^{n-1}\right| = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = \frac{2}{3}$

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is a convergence series

But $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ is a divergence series

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is a conditionally convergence series.

Proposition (3.9):

If $\sum_n a_n$ is absolutely convergence series, then $\sum_n a_n$ is a convergence series.

Proof: let $< S_n >$ be sequences of partial sum of $\sum_n a_n$ and $< T_n >$ be sequences of partial sum of $\sum_n |a_n|$.

$< T_n > = \sum_{i=1}^{n}|a_i| = |a_1| + |a_2| + |a_3| + \cdots + |a_n|$

$< T_m > = \sum_{i=1}^{m}|a_i| = |a_1| + |a_2| + |a_3| + \cdots + |a_m|$

$a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + a_{n+2} + \cdots + a_{n+t}$

In particular take $m = n + t$

$\forall \in > 0 \;\; \forall\, n, m \in Z^+ \quad n < m$

$|T_m - T_n| = |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+t}| \quad \forall\, n \in N$

$\because \sum_n |a_n|$ is a convergence series, then $\exists\, k \in Z^+ \; s.t$

$|T_m - T_n| = |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+t}| < \in \;\; \forall\, n > k \;\; \forall\, t > 1$

$|S_m - S_n| = |a_{n+1} + a_{n+2} + \cdots + a_{n+t}|$

$\leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+t}| < \in \;\; \forall\, n > k \;\; \forall\, t > 1$

$\therefore < S_n >$ is a Cauchy sequence, hence $< S_n >$ is a convergence sequence (**R** is complet).

Remark:

The converse of proposition (3.9) is not true in general as the following example show:-

Example:

$\sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{n}$

4-The Root Test:

If $\sum_n a_n$ is a series such that $\forall \in > 0 \ \exists \ k = k(\in)$ and a positive real number b with $\sqrt[n]{a_n} < b < 1 \quad \forall n > k$, then $\sum_n a_n$ is a convergence series.

5-The Ratio Test:

Let $\sum_n a_n$ be a series with $a_n > 0$, if $\forall \in > 0 \ \exists \ k = k(\in)$, $b \in \boldsymbol{R}$, $b > 0$ such that $\frac{a_{n+1}}{a_n} < b < 1 \quad \forall n > k$. Then $\sum_n a_n$ is a convergence series.