Countable sets

Q is countable set

Proposition (2.19):

R is not countable set.

Proof: Let $S = \{a_1, a_2, \dots, a_n, \dots\} \subseteq R$ be a countable set $(S \neq R)$?

Let I_1 be a closed interval in R such that $|I_1| < 1$ and $a_1 \notin I_1$.

Let I_2 be a closed interval in R such that $|I_2| < \frac{1}{2}$ and $a_2 \notin I_2$ and $I_1 \supseteq I_2$.

$\vdots$

Let I_n be a closed interval in R such that $|I_n| < \frac{1}{n}$ and $a_n \notin I_n$ and $I_{n-1} \supseteq I_n$.

$I_1, I_2, I_3, \cdots, I_n, \cdots, \quad |I_n| \longrightarrow 0$

$\left|\frac{1}{n}\right| \longrightarrow 0$ by nested theorem

$\cap_n I_n = \{y\} \quad y \in R$

$y \in I_n \ \forall\, n$ and $y \neq a_n \ \forall\, n$, $\therefore \ y \notin S$, $\therefore \ S \neq R$

Corollary (2.20):

The set of irrational number is uncountable set.

(The union of two countable set is countable)

Proof: If not, then $R = Q \cup Q' \Rightarrow$ countable C!

$\therefore$ Q' is not countable.

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Chapter (3)

The infinite series

Definition(3.1):

Let $< a_n >$ be a sequence of real numbers the sum $\sum_{i=1}^{\infty} a_i$ is called an infinite series

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$\vdots$

$$S_n = \sum_{i=1}^{n} a_i$$

Definition(3.1):

$< S_n >$ is called the sequence of partial sum

If $< S_n >$ converges to S, then we say that $\sum_{i=1}^{\infty} a_i$ is a converges series, in this case we write $\sum_{i=1}^{\infty} a_i = S$ and if $< S_n >$ is a divergence sequence, then we say that $\sum_{i=1}^{\infty} a_i$ is a divergence series.

Some Types of Series

1) ***Geometric series:***

A series in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

r is called the base of the series.

$$S_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

$$(1-r)\,S_n = (1-r)(a + ar + ar^2 + \cdots + ar^{n-1})$$

$$= a - ar^n$$

$$= a(1-r^n)$$

If $r \neq 1$, then $S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a}{(1-r)} - \frac{ar^n}{(1-r)}$ $\quad |r| < 1$

$\therefore\ r^n \to 0$ when $|r| < 1$ $\quad i.e\ (-1 < r < 1)$

$S_n \to \frac{a}{(1-r)}$ when $|r| < 1$

When $|r| > 1$, then $< r^n >$ is not bounded, hence $< S_n >$ is not bounded.

$\therefore\ S_n$ is a divergence sequence, hence the series is divergence.

When $r = 1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} a = a + a + a + \cdots + a + \cdots$

$S_n = < na >$ not bounded sequence, hence not converges sequence.

$\therefore\ \sum_{n=1}^{\infty} ar^{n-1}$ is divergence when $r = 1$.

When $r = -1$

$$\therefore \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} a(-1)^{n-1}$$

$$= a - a + a - a + a + \cdots + a(-1)^{n-1} + \cdots$$

$S_n = 0$ when n is even $i.e\ S_{2m} = 0$ $\quad m \in \mathbb{Z}^+$, $S_n = \sum_{i=1}^{n} a_i$

$S_{2m+1} = a$ $\quad$ *when n is odd*

$|S_{2m+1} - S_{2m}| = |a| < \epsilon$?

$\therefore < S_n >$ is not a Cauchy sequence, hence $< S_n >$ is a divergence sequence, $i.e\ \sum_{n=1}^{\infty} ar^{n-1}$ is a divergence series when $r = -1$.

Examples(3.2):

1. $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)^{n} = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^{n-1}$

$-1<\frac{1}{4}<1 \quad = \sum_{n=1}^{\infty}\left(-\frac{1}{8}\right)\left(\frac{1}{4}\right)^{n-1}$

$$= \frac{\left(-\frac{1}{8}\right)}{\left(1-\frac{1}{4}\right)}.$$

2. $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)(-0.1)^{n-1} = \frac{\frac{1}{3}}{1-(-0.1)}$

2) *Harmonic series:*

Is of the form $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergence series.

Let $S_1 = 1$

$$S_2 = 1+\frac{1}{2}$$

$$S_3 = 1+\frac{1}{2}+\frac{1}{3}$$

$\vdots$

$$S_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = \sum_{k=1}^{n}\frac{1}{k}$$

$\vdots$

$$S_{2n}-S_n = \left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right] - \left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right]$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = n\,\frac{1}{2n} = \frac{1}{2}$$

$$|S_{2n} - S_n| \geq \frac{1}{2} > \in = \frac{1}{4}$$

$\therefore < S_n >$ is not a Cauchy sequence, $< S_n >$ is not a converges sequence, hence $< S_n >$ is a divergence sequence.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergence series (has no sum).

3) *The Alternating series:*

Is of the form $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, where $< a_n >$ is a decreasing sequence and $a_n \to 0$, $a_n > 0 \ \forall n$, the alternating series is a converges series.

To show that:

When n is even, $n = 2m$, $m \in Z^+$

Let $< S_n >$ be the sequence of partial sums

$$\begin{aligned} S_{2m} &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2m-1} - a_{2m} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}) \\ &= (a_1 - a_{2m}) - (a_2 - a_3) - \cdots - (a_{2m-2} - a_{2m-1}) \end{aligned}$$

$(a_{i-1} - a_i) > 0$

$S_{2m} \leq a_1 - a_{2m} \leq a_1 \quad \forall m$ $\quad a_1$

$\therefore |S_{2m}| \leq a_1 \quad \forall m$ so it is bounded also S_{2m} is increasing

$\therefore < S_n >$ is a convergence sequence.

If n is odd, $n = 2m + 1$, $m \in Z^+$

$$\begin{aligned} S_{2m+1} &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2m-1} - a_{2m} + a_{2m+1} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}) + a_{2m+1} \\ &= (a_1 - a_{2m}) - (a_2 - a_3) - \cdots - (a_{2m-2} - a_{2m-1}) + a_{2m+1} \end{aligned}$$

$S_{2m+1} \leq a_1 \quad \forall m$

$\therefore |S_{2m+1}| \leq a_1 \quad \forall m$ so it is bounded by a_1.

$< S_{2m+1} >$ is a bounded monotonic sequence

$\therefore < S_{2m+1} >$ is a converges sequence.

Claim: $< S_{2m} >$ and $< S_{2m+1} >$ has the same limit point.

$|S_{2m+1} - S_{2m}| = |a_{2m+1}|$

$\because \ a_n \to 0$ when $n \to \infty$ (given) $\therefore a_{2m+1} \to 0$

$$\begin{array}{ccc} S_{2m+1} & - & S_{2m} \\ \downarrow & & \downarrow \\ a & & b \end{array} \to 0 \quad \Rightarrow \quad |S_{2m+1} - S_{2m}| \to a - b$$

$\therefore a - b = 0 \qquad \Rightarrow \qquad a = b$

$\therefore < S_{2m} >$ and $< S_{2m+1} >$ have the same limit point.

Example:

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ Alternating series is a converges series.

$\frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \cdots$ decreasing sequence $\frac{1}{n} \to 0$ when $n \to \infty$ and

$\frac{1}{n} > 0 \Rightarrow$ Alternating series $\Rightarrow$ converges series.

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4) $\forall k$ *we have* $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{(k+1)}$

Let $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{1\cdot(2)} + \frac{1}{2\cdot(3)} + \frac{1}{3\cdot(4)} + \cdots + \frac{1}{n(n+1)} + \cdots$

$$S_n = \sum_{i=1}^{n}\frac{1}{i(i+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)}$$

$$= \sum_{i=1}^{n}\frac{1}{i} - \frac{1}{i+1}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n}{n+1}$$

$S_n = \langle\frac{n}{n+1}\rangle \to 1$ converges sequence,

$\therefore \sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$

Proposition (3.3):

Let $\sum_n a_n$ and $\sum_n b_n$ be two convergence series, if $\sum_n a_n = S$ and $\sum_n b_n = T$. Then $\sum_n a_n + \sum_n b_n = \sum_n (a_n + b_n)$ is a convergence series and $\sum_n a_n + \sum_n b_n = S + T$.

Proof: let $< S_n >$ and $< T_n >$ be sequences of partial sum of $\sum_n a_n$ and $\sum_n b_n$ respectively.

$\sum_n a_n = S$ and $\sum_n b_n = T$

$\therefore\ S_n \to S$ and $T_n \to T$

Let V_n be a sequence of partial sum of $\sum_n a_n + \sum_n b_n$.

$V_n = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

$= \underbrace{a_1 + a_2 + \cdots + a_n}_{S_n} + \underbrace{b_1 + b_2 + \cdots + b_n}_{T_n} \to S + T$

$\therefore\ V_n \to S + T$ i.e V_n converges seq.

$\therefore\ \sum_n a_n + \sum_n b_n = S + T$

Corollary (3.4):

If $\sum_n a_n$ is a convergence series and $\sum_n b_n$ is a divergence series, then $\sum_n a_n + \sum_n b_n$ is a divergence series.

Proof: suppose the result is not true; i.e $\sum_n a_n + \sum_n b_n$ is a convergence series.

$\because$ $\sum_n a_n$ is a convergence series, then $-\sum_n a_n$ is a convergence series; ? let $\langle S_n \rangle$ be sequence of partial sum of $\sum_n a_n$, then $S_n \to S$. let $\langle T_n \rangle$ be sequence of partial sum of $c\sum_n a_n$, then $T_n \to T$

$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$

$T_n = c \sum_{i=1}^{n} a_i = c(a_1 + a_2 + \cdots + a_n) = c\, S_n$

$S_n \to S$, $\underbrace{cS_n \to cS}_{T_n \to T}$ $\Rightarrow c\sum a_n$ conv.

$\therefore -\sum_n a_n$ is a convergence series.

$(\sum_n a_n + \sum_n b_n) + (-\sum_n a_n) = \sum_n b_n$ convergence series by (3.3)

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$\therefore$ $\sum_n a_n + \sum_n b_n$ is a divergence series.

H.W: Give an example of two divergence series such that the sum of them is convergence.

Some Kinds of Test:

1-The Convergence Test:

Proposition (3.5):

If $\sum_n a_n$ is a convergence series, then the sequence $< a_n >$ is converge to zero.

Proof: let $< S_n >$ be sequence of partial sum of $\sum_n a_n$

$\because$ $\sum_n a_n$ is a convergence series, then $< S_n >$ is a convergence sequence and hence $< S_n >$ is a Cauchy sequence.

$i.e\ \forall \in > 0\ \exists\, k \in Z^+\ \ k = k(\in)$ s.t $|S_n - S_m| < \in\ \ \forall\, n, m > k.$

In particular put $m = n + 1$

$|S_n - S_m| = |-a_{n+1}| = |a_{n+1}| = |a_{n+1} - 0| < \in\ \ \forall\, n > k$

$< a_n >$ converge to zero.

* If $<a_n> \not\to 0 \Rightarrow \sum_n a_n$ is diverge

Example:

$< \frac{n}{2n+1} >$ is converge to $\frac{1}{2}$

$\therefore$ $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ is a diverges series, since not converge to zero.

Remark:

In general the converse of proposition (3.5) is not true.