Chapter (2)

The sequences of real numbers

Definition(2.1) :-

Let $f: N \to R$ be a function, then $f(n) = a_n \quad \forall n \in Z$, is called a sequence of real numbers which will be denoted by $\langle a_n \rangle$ or $\{a_n\}$.

$\langle a_n \rangle = a_1, a_2, \ \dots \ , a_n, \dots$

Example:- $\langle \frac{1}{n} \rangle = 1 \, , \frac{1}{2} \, , \frac{1}{3} \, , \frac{1}{4} \, , \dots , \frac{1}{n} \, , \ \dots$

$$\langle \frac{1}{2^n} \rangle = \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \dots, \quad \frac{1}{2^n}, \quad \dots$$

$$\langle (-1)^n \rangle = -1, 1, -1, \dots \ , (-1)^n, \dots$$

$$\langle 3^n \rangle = 3, 9, 81, \dots, 3^n, \quad \dots$$

$$\langle \frac{1}{2} \rangle = \frac{1}{2}, \quad \frac{1}{2}, \quad \frac{1}{2}, \dots, \quad \frac{1}{2}, \quad \dots$$

$$\langle \frac{n}{n+1} \rangle = \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \dots, \quad \frac{n}{n+1}, \quad \dots$$

Converging sequences:

Definition(2.2) :-

Let $\langle a_n \rangle$ be a sequence of real numbers, we say that $\langle a_n \rangle$ is converging sequence if there exists a real number a_0 satisfies for all $\in > 0$, $(0 < \in < 1)$ there exist a positive integer $k = k(\in)$ (depend on $\in$) such that $|a_n - a_0| < \in \quad \forall \, n > k$.

i.e if $a_n \to a_0$, then $\lim_{n \to \infty} a_n = a_0$.

Otherwise the sequence is divergence.

Proposition (2-3):-

If the sequence $\langle \mathbf{a_n} \rangle$ is convergence sequence, then the limit point is unique.

Proof: Suppose that $\mathbf{a_n} \to \mathbf{a_0}$ and $\mathbf{a_n} \to \mathbf{b_0}$ and $\mathbf{a_0} \neq \mathbf{b_0}$, then $\mathbf{0} < \boldsymbol{d} = |\mathbf{a_0} - \mathbf{b_0}|$.

$\because \; a_n \to a_0$

$\forall \; \epsilon > 0$, in particular take $\epsilon = \frac{d}{2}$, $\exists \; k_1 \left(\frac{d}{2}\right)$ such that

$|a_n - a_0| < \frac{d}{2} \quad \forall n > k_1$.

$\because \; a_n \to b_0$

$\forall \; \frac{d}{2} > 0, \exists \; k_2 \left(\frac{d}{2}\right)$ such that $|a_n - b_0| < \frac{d}{2} \quad \forall n > k_2$.

$0 < d = |a_0 - b_0| = |a_0 - a_n + a_n - b_0|$

$\leq |a_n - a_0| + |a_n - b_0|$

$< \frac{d}{2} + \frac{d}{2} = d \; C! \;\; (d < d) \;, \quad \forall n > k = \max\{k_1, k_2\}$.

Examples:-

1) Is $\langle \frac{1}{n} \rangle$ converge to 0 i.e $\lim_{n\to\infty} \frac{1}{n} = 0$

$\langle \frac{1}{n} \rangle = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$

Let $\epsilon > 0$, to find $k(\epsilon)$ such that:

$\left|\frac{1}{n} - 0\right| < \epsilon \quad \forall \; n > k$.

Proof: $\left|\frac{1}{n}\right| = \frac{1}{n}$, since $n \in Z^+$.

By Archimedean $\forall \; \epsilon > 0, \; \exists \; k \in Z^+ \; s.t \; \frac{1}{k} < \epsilon$

$\frac{1}{n} < \frac{1}{k} < \epsilon \quad \forall \; n > k$

$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{k} < \epsilon \quad \forall \; n > k$.

$\therefore \left|\frac{1}{n} - 0\right| < \epsilon \quad \forall n > k$

2) Is $\langle a_n \rangle = \langle 3 \rangle$ converge to 3, $\lim_{n\to\infty} 3 = 3$

$f: N \to R, \quad f(n) = a_n = 3\,, \; , \; \langle 3 \rangle = 3,3,3, \ldots$

$\forall\ \epsilon > 0,\ \exists\, k = 0\ ,\ |3 - 3| = 0 < \epsilon \quad \forall n > 0.$

3) Let $\langle a_n \rangle$ be define by:

$$a_n = \begin{cases} -2 & n > 10^7 \\ n & n \le 10^7 \end{cases}$$

$\langle a_n \rangle = 1, 2, 3, 4, 5, \ldots, 10^7, -2, -2, \ldots$

This sequence convergence to (-2).

$\forall\ \epsilon > 0, \quad \exists\, k = 10^7\ ,\ |a_n - (-2)| < \epsilon \quad \forall n > 10^7.$

4) Let $\langle a_n \rangle = \langle (-1)^n \rangle$ be a divergence sequence.

$\langle (-1)^n \rangle = -1, 1, -1, 1, \ldots$

If $a_0 = -1$, then for all $\epsilon > 0$, $(-1-\epsilon, -1+\epsilon)$ contain all odd terms but doesn't contain any even term and since the even terms are infinite, then $a_n \nrightarrow -1$.

If $a_0 = 1$, then for all $\epsilon > 0$, $(1-\epsilon, 1+\epsilon)$ contain all even terms but doesn't contain any odd term and since the odd terms are infinite, then $a_n \nrightarrow 1$.

If $a_0 \neq 1$ or $a_0 \neq -1$

$0 < d_1 = |a_0 - 1|\ ,\ 0 < d_2 = |a_0 - (-1)|.$

If we choose $\epsilon \le \min\{d_1, d_2\}$, then any open interval $(a_0-\epsilon, a_0+\epsilon)$ doesn't contain any term of the sequence and hence $a_n \nrightarrow a_0$.

$\therefore \langle (-1)^n \rangle$ is a divergence sequence.

<u>**H.W**</u>: Which of the following sequence convergence or divergence.

1. $\langle \frac{n}{n+1} \rangle$.
2. $\langle \frac{1}{2^n} \rangle$.
3. $\langle 3^n \rangle$.

Bounded sequences:

Definition(2.4) :-

A sequence $\langle a_n \rangle$ of real numbers is said to be a bounded sequence, if there exists a real number M such that $|a_n| \leq M \quad \forall n$.

$i.e \; -M \leq a_n \leq M$.

Examples:-

1) $a_n = \langle \frac{1}{n} \rangle$ is bounded sequence since $-1 < 0 \leq \frac{1}{n} \leq 1$.
2) $a_n = \langle 3 \rangle$ is bounded sequence since $-3 \leq 3 \leq 3$.
3) $\langle a_n \rangle = \begin{cases} -2 & n > 10^7 \\ n & n \leq 10^7 \end{cases}$

$\langle a_n \rangle = 1, 2, 3, 4, 5, \ldots, 10^7, -2, -2, -2, \ldots$
This sequence is bounded since $-10^7 \leq a_n \leq 10^7$.

4) $\langle a_n \rangle = \langle (-1)^n \rangle = -1, 1, -1, 1, \ldots$ is bounded sequence since $-1 \leq a_n \leq 1$.
5) $\langle 2^n \rangle = 2, 4, 8, 16, \ldots, 2^n, \ldots$ is not bounded sequence since $0 < 2^n < ?$. (bounded below but not bounded above).

Proposition (2-5):-

Every convergence sequence is a bounded sequence.

Proof: Let $\langle \mathbf{a_n} \rangle$ be a convergence sequence, that convergence to $\mathbf{a_0}$

$i.e \; a_n \to a_0$

$\forall \in > 0, \exists k = k(\in)$ such that $|a_n - a_0| < \in < 1 \quad \forall n > k$.

$|a_n| - |a_0| \leq |a_n - a_0| < 1 \quad \forall n > k$.

$\Rightarrow |a_n| - |a_0| \leq 1 \quad \forall n > k$

$\therefore |a_n| \leq |a_0| + 1 \quad \forall n > k$.

$|a_1|, |a_2|, \ldots, \underline{|a_k|, \quad |a_{k+1}|, \quad |a_{k+2}|, \ldots}$

$\leq |a_0| + 1$.

ake $M = \{ |a_1|, |a_2|, \dots, |a_k|, \dots, |a_0| + 1 \}$.
$\therefore |a_n| \leq M \quad \forall n$.

Example:-

$\langle 2^n \rangle = 2, 4, 8, 16, \dots, 2^n, \dots$ is not bounded sequence and by this theorem is divergence.

Remark(2.6):-

The converse of proposition (2.5) is not true in general, as the following example shows.

Example:-

$\langle (-1)^n \rangle$ is bounded sequence which is a divergence sequence.

Monotonic sequences:

Definition(2.7) :-

Let $\langle a_n \rangle$ be a sequence, we say that $\langle a_n \rangle$ is a non- decreasing sequence, if $a_n \leq a_{n+1} \quad \forall n$.

$\langle a_n \rangle$ is an increasing sequence, if $a_n < a_{n+1} \quad \forall n$.

$\langle a_n \rangle$ is a non- increasing sequence, if $a_n \geq a_{n+1} \quad \forall n$.

And $\langle a_n \rangle$ is a decreasing sequence, if $a_n > a_{n+1} \quad \forall n$.

And we say that $\langle a_n \rangle$ is a monotonic sequence, if $\langle a_n \rangle$ satisfies one of the above conditions.

Examples:-

1) $\langle \frac{1}{n} \rangle = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$ is decreasing sequence.
2) $\langle \frac{n}{n+1} \rangle = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is an increasing sequence.
3) $\langle 3 \rangle = 3, 3, 3, \dots, 3, \dots$ is a non- increasing sequence and a non- decreasing sequence.

$\langle(-1)^n\rangle = -1, 1, -1, 1, \ldots$ is not monotonic sequence.

Proposition (2-8):-

Every bounded monotonic sequence is convergence sequence.

Proof: Let $\langle a_n \rangle$ be a sequence in R, $\because$ $\langle a_n \rangle$ is bounded sequence.

$\therefore\ \exists M$, such that $|a_n| \leq M \quad \forall n$.

$S = \{a_n : n \in N\}$ bounded (above and below).

(1) Suppose $\langle a_n \rangle$ is a non- decreasing sequence,

Since S is bounded above, then by completeness of real number S has a least upper bound say y.

$y = \sup(S) = l.u.b(S) \qquad a_n \leq y \qquad \forall n \in N.$

Claim: $a_n \to y$

$y - \frac{\epsilon}{2} < y \qquad \therefore\ y - \frac{\epsilon}{2}$ is not an upper bound.

$\exists\, k \in Z^+$ such that $a_k > y - \frac{\epsilon}{2}$

$$y - \frac{\epsilon}{2} < a_k \leq a_n < y + \frac{\epsilon}{2}$$

$$y - \frac{\epsilon}{2} < a_n < y + \frac{\epsilon}{2} \qquad \forall n > k$$

$$|a_n - y| < \frac{\epsilon}{2} \qquad \forall n > k\ .$$

(2) Suppose $\langle a_n \rangle$ is a non- increasing sequence,

i.e $\exists M$, such that $|a_n| \leq M \quad \forall n$.

Since S is bounded below, where $S = \{a_n : n \in N\} \subseteq R$, then by completeness of real number S has greatest lower bound, say a_0.

$a_0 = \inf(S) = g.l.b(S) \qquad a_n \geq a_0 \qquad \forall n \in N.$

Claim: $a_n \to a_0$ ($\forall \in > 0, \exists k \in Z^+$ such that $|a_n - a_0| < \in$ $\forall n > k$).

$a_0 = \inf(S) = g.l.b(S)$ $\quad a_n \leq a_0 \quad \forall n \in N$ ...(1)

$a_0 + \in$ is not a lower bound (since $a_0 < a_0 + \in$)

$\therefore \exists k \in Z^+$ such that $a_k < a_0 + \in$ ...(2).

Since $\langle a_n \rangle$ is not increasing sequence, then $a_n \leq a_k$ ... (3)

From (1), (2), (3) $a_0 - \in < a_n \leq a_k < a_0 + \in$

$a_0 - \in \leq a_n \leq a_0 + \in \qquad \forall n > k$

$\Rightarrow |a_n - a_0| < \in \qquad \forall n > k$.

$\therefore \langle a_n \rangle$ is converges.

Examples:-

1) $\langle \frac{1}{n} \rangle = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$.

$S = \left\{ \frac{1}{n} : n \in N \right\}$.

$= \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots \right\}$.

This sequence is decreasing and bounded (below, above).

$a_n \to g.l.b(S) = \{0\}$.

2) Converges $\nRightarrow$ monotonic.

Let $\langle a_n \rangle = \begin{Bmatrix} n & n \leq 10^2 \\ -1 & n > 10^2 \end{Bmatrix}$.

$= 1, 2, 3, 4, 5, \dots, 10^2, -1, -1, -1, \dots$.

It is converges but not monotonic sequence.