Remarks(4.5):

Let (X, d) be a metric space, then

1) For any $x, y, z \in X$, we have .

$$|d(x,z) - d(z,y)| \leq d(x,y)$$

2) For any $x_1, x_2, \cdots, x_n \in X$, we have

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$$

Proof(1):

$d(x,z) \leq d(x,y) + d(y,z) \qquad \cdots (1)$

$d(z,y) \leq d(z,x) + d(x,y) \qquad \cdots (2)$

From (1) we get: $d(x,z) - d(y,z) \leq d(x,y)$

From (2) we get: $-d(x,y) \leq d(z,x) - d(z,y)$

$\therefore \ |d(x,z) - d(z,y)| \leq d(x,y)$

Proof(2):

By induction on the element of X.

$n = 3$

$x_1, x_2, x_3 \in X$, then

$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3) \qquad \cdots (3)$

Suppose the result is true for any $k = n - 1 < n$

i.e

$$d(x_1, x_{n-1}) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-2}, x_{n-1})$$

To prove is true for any n

$$d(x_1, x_n) \leq d(x_1, x_{n-1}) + d(x_{n-1}, x_n) \quad \text{by (3)}$$
$$\leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)$$

Basic principles of topology:

Definition(4.6):

Let (X, d) be a metric space, and $x_0 \in X$, $r \in R$, $r > 0$, then:

$$B_r(x_0) = \{\, x \in X : d(x, x_0) < r\}$$

Is called a ball of radius r and center x_0.

$$D_r(x_0) = \{\, x \in X : d(x, x_0) \leq r\}$$

Is called a disk of radius r and center x_0.

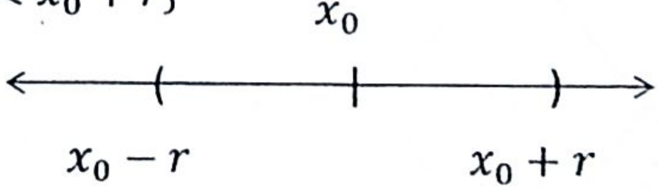
Examples:

1) (R, d) is a metric space.

$B_r(x_0) = \{\, x \in R : |x - x_0| < r\}$

$= \{\, x \in R : x_0 - r < x < x_0 + r\}$

$= (x_0 - r\,,\, x_0 + r)$



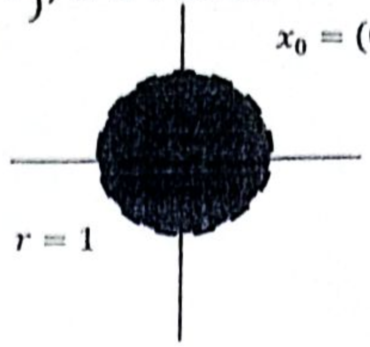
$D_r(x_0) = \{\, x \in R : |x - x_0| \leq r\}$

$= [x_0 - r\,,\, x_0 + r]$

2) (R^2, d) is a metric space

$B_r(x_0) = \left\{x \in R^2 : \sqrt{(x-x_0)^2 + (y-y_0)^2} < r\right\}$; d is a usual distance

$= \{x \in R^2 : (x-x_0)^2 + (y-y_0)^2 < r^2\}$



(R^n, d) is a metric space; d is a usual distance

$$B_r(x_0) = \left\{\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_n) \in \mathrm{R}^2 : \sqrt{(\mathrm{x}_1-\mathrm{x}_0)^2 + (\mathrm{x}_2-\mathrm{x}_0)^2 + \cdots + (\mathrm{x}_n-\mathrm{x}_0)^2} < \mathrm{r}\right\}$$

3) (R^2, d) is a metric space

Where $d: R^2 \times R^2 \to R$ defined by $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

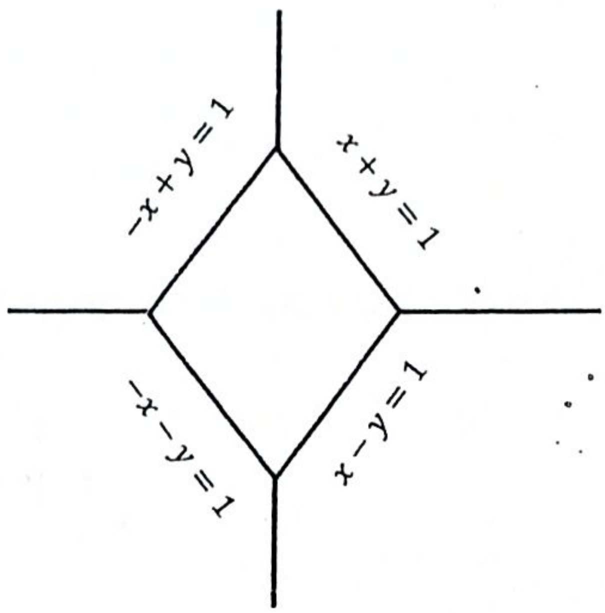
$x = (x_1, x_2)$, $\quad y = (y_1, y_2)$

$B_1(0) = \{ (x,y) \in R^2 : |x-0| + |y-0| < 1\}$

$B_1(0,0) = \{ (x,y) \in R^2 : |x| + |y| < 1\}$

We have the following cases:

$x, y > 0$	$x + y = 1$
$x < 0, y > 0$	$-x + y = 1$
$x > 0, y < 0$	$x - y = 1$
$x < 0, y < 0$	$-x - y = 1$



<u>Definition(4.7):</u>