Cauchy sequences:

Definition(2.10) :-

A sequence $\langle a_n \rangle$ is called a Cauchy sequence if $\forall \in > 0$ there exist a positive integer $k = k(\in)$ such that $|a_n - a_m| < \in \quad \forall n, m > k$.

Proposition (2-11):-

Every convergence sequence in R or Q is a Cauchy sequence.

Proof: Let $\langle a_n \rangle$ be a convergence sequence, that convergence to a_0

i.e $a_n \to a_0$

$\forall \in > 0, \exists\, k = k(\in)$ such that $|a_n - a_0| < \frac{\in}{2} \qquad \forall\, n > k$.

$$|a_n - a_m| = |a_n - a_0 + a_0 - a_m| .$$

$$\leq |a_n - a_0| + |a_m - a_0|$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in \qquad \forall\, n > k, \forall\, m > k .$$

$\therefore\ |a_n - a_m| < \in \qquad \forall\, n, m > k$.

Remark (2-12):-

The converse of Proposition (2-11) is not true in general in the field of rational number.

We need the following lemma:

Lemma (2-13):-

For any real number r, there exists a sequence of rational number converge to r.

Proof: Let $r \in R \qquad\qquad r - 1 < r + 1$

By the density of rational numbers $\exists\, r_1 \in Q$ such that

$r-1 < r_1 < r+1 \Rightarrow r-\frac{1}{2} < r+\frac{1}{2}$

And by the density of rational numbers

$\exists\, r_2 \in Q$ *such that* $r-\frac{1}{2} < r_2 < r+\frac{1}{2}$

Continue in this way we get a sequence of rational numbers $< r_n >$

$$r-\frac{1}{n} < r_n < r+\frac{1}{n} \qquad \forall n \in N \qquad \dots(*)$$

Claim : $r_n \to r$ *from* $(*)$ $|r_n - r| < \frac{1}{n}$

(Arch.) $\forall \in > 0$, $\exists\, k = k(\in)$ *such that* $\frac{1}{k} < \in$

$|r_n - r| < \frac{1}{n} < \frac{1}{k} < \in \quad \forall n > k \quad (\forall\, n > k \to \frac{1}{n} < \frac{1}{k})$

$\therefore |r_n - r| < \in$

$\therefore r_n \to r$

Remark (2.12) :-

The converge of proposition (2.11) in general is not true in Q

Proof: *let* $r = \sqrt{2} \notin Q$

$\therefore$ by lemma (2.13) , $\exists$ a sequence of rational numbers

$< r_n >$ such that : $r_n \to \sqrt{2}$

$\because r_n \to \sqrt{2}$ $\therefore$ by proposition (2.11) $< r_n >$ is a Cauchy sequence, in R (since converg in R,)
but $< r_n >$ is not converges in Q

H.W:

(1) $< \frac{1}{n} >$, $R - \{0\}$

(2) For any real number there exists a sequence of irrational numbers converge to r.

Theorem (2.14):- (The nested intervals theorem)

Let $< I_n >$ be a sequence of closed intervals such that $I_{n+1} \subseteq I_n \quad \forall n$. Then $\cap_n \; I_n \neq \emptyset$.

Moreover if $< |I_n| >$ converges to zero, then $\cap_n \; I_n$ consists of only one point.

Proof: Let $< I_n > = [a_1, b_1], [a_2, b_2], \dots, [a_n, b_n], \dots$

Let $S_1 = \{a_1, a_2, \dots, a_n, \dots\}$

$S_2 = \{b_1, b_2, \dots, b_n, \dots\}$

$\because \forall \; n, m \quad I_{n+1} \subseteq I_n \Rightarrow if \;\; n \le m \;\Rightarrow\; a_n \le a_m, \qquad b_m \le b_n$

If $\; n > m \;\Rightarrow\; a_m < a_n < b_n < b_m$

$\therefore \; a_n < b_n$

$\therefore$ Each element in S_2 is an upper bound of S_1

$\therefore \; S_1$ is bounded above

$\therefore$ by completeness of real numbers S_1 has a least upper bound say y

$y = sup(S_1)$

$\therefore \; a_n \le y \qquad \forall n \in N$ and $\; y \le b_n \qquad \forall n \in N \;\; (y = l.u.b(S_1))$

$a_n \le y \le b_n \qquad \forall n$

$\therefore y \in \cap_n I_n \;\;\Rightarrow \cap_n \; I_n \neq \emptyset$

-If $< |I_n| > \longrightarrow 0$

Suppose, there exists another point z, such that

$z \in \cap_n I_n \quad and \quad y \neq z$

$0 < d = |y - z|$

$\because < |I_n| > \longrightarrow 0 \quad \therefore \exists\ k \in Z^+$ such that $|I_k| < d$

$0 < d = |y - z| \leq |I_k| < d \quad C!$

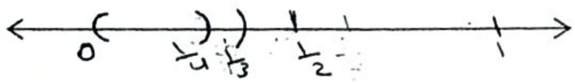
$\therefore\ y = z.$

Remark (2.15):

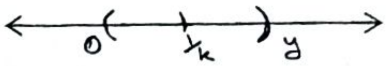
In general theorem (2.14) is not true if the interval is not closed. As the following example show:

Example: $I_n = \left(0, \frac{1}{n}\right) \quad \forall n$

$\cap_n\ I_n = \emptyset$?



If $\cap\ I_n = \{y\}$



$\forall\, y > 0\,,\ \exists\, k \in Z^+ \quad s.t \qquad 0 < \frac{1}{k} < y \quad C!\,?$

$i.e\ \ y \notin I_k = \left(0, \frac{1}{k}\right)$

$\therefore\ \cap_n I_n = \emptyset\,.$

Completeness of real numbers

Every Cauchy sequence in R is converging in R.

Proposition (2.17):

Every Cauchy sequence is a bounded sequence.

Proof: let $< a_n >$ be a Cauchy sequence, **i.e** $\forall \in > 0\,, \exists\, k = k(\in)$ such that $|a_n - a_m| < \in \quad \forall\ n, m > k$

In particular take $m = k + 1$

$|a_n| - |a_{k+1}| \leq |a_n - a_{k+1}| < \in < 1 \quad \forall n > k$

$\therefore |a_n| < |a_{k+1}| + 1 \qquad \forall n > k$

Take $M = \max\{|a_{k+1}| + 1, |a_1|, |a_2|, \dots, |a_k|\}$

$|a_n| \leq M \quad \forall n$.

Proposition (2.18):

Let $< a_n >$ and $< b_n >$ be two convergence sequences such that $a_n \to a_0$ and $b_n \to b_0$, then:

1) $a_n \pm b_n \to a_0 \pm b_0$.
2) $a_n \cdot b_n \to a_0 \cdot b_0$
3) $c \cdot a_n \to c \cdot a_0 \quad \forall c \in R$
4) $\frac{a_n}{b_n} \to \frac{a_0}{b_0} \quad b_n \neq 0 \ \forall n, \ \ b_0 \neq 0$.

Proof: (4) $\because \ a_n \to a_0$

$\therefore \forall \epsilon > 0, \exists\, k_1 = k_1\left(\frac{\epsilon}{2}\right)$ such that $|a_n - a_0| < \frac{\epsilon |b_0|}{2} \quad \forall\, n > k_1$

$\because \ b_n \to b_0$

$\therefore \ \exists\, k_2 = k_2\left(\frac{\epsilon}{2}\right)$ such that $|b_n - b_0| < \frac{\epsilon M_2 |b_0|}{2 M_1} \quad \forall n > k_2$

bounded

$\because < a_n >$ is converge, $\therefore \ \exists\, M_1$ s.t $|a_n| \leq M_1 \quad \forall n$

$\because < b_n >$ is converge, $\therefore \ \exists\, M_2$ s.t $|b_n| \leq M_2 \quad \forall n$

$$\left|\frac{a_n}{b_n} - \frac{a_0}{b_0}\right| = \left|\frac{b_0 a_n - a_n b_n + a_n b_n - a_0 b_n}{b_n\, b_0}\right|$$

$$\leq \frac{|a_n|\,|b_n - b_0|}{|b_n|\,|b_0|} + \frac{|b_n|\,|a_n - a_0|}{|b_n|\,|b_0|}$$

$$< \frac{M_1}{M_2} \cdot \frac{\epsilon\, |b_0| M_2}{2 M_1 |b_0|} + \frac{\epsilon\, |b_0|}{2|b_0|} \quad \forall\, n > k_1$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon \quad \forall n > k = \max\{k_1, k_2\}.$$