Definition (4.21):

Let (X, d) be a metric space and $\langle x_n \rangle$ be a sequence in X, we say that $\langle x_n \rangle$ is a convergence sequence if there exists $x_0 \in X$ such that $\forall \epsilon > 0, \exists\, k = k(\epsilon)$ satisfise:

$$d(x_n, x_0) < \epsilon \qquad \forall\, n > k$$

i.e any ball with center x_0 and radius ϵ contain most of the terms of the sequence.

Proposition (4.22):

If $\langle x_n \rangle$ is a convergence sequence in X that converges to x_0, then x_0 is unique.

Proof: Suppose there exists another limit point y_0 for $\langle x_n \rangle$

i.e $x_n \to y_0$ and $x_0 \neq y_0$.

$0 < d = d(x_0, y_0)$ take $\epsilon = \frac{1}{2}d$

$\therefore \exists\ B_{\frac{1}{2}d}(x_0)$ and $B_{\frac{1}{2}d}(y_0)$ such that $B_{\frac{1}{2}d}(x_0) \cap B_{\frac{1}{2}d}(y_0) = \emptyset$

$\because x_n \to x_0$ and $x_n \to y_0$, then each of balls $B_{\frac{1}{2}d}(x_0)$ and $B_{\frac{1}{2}d}(y_0)$ contain most of the term of the sequence but $B_{\frac{1}{2}d}(x_0) \cap B_{\frac{1}{2}d}(y_0) = \emptyset$ a contradiction.

$\therefore x_n \to y_0$

Definition (4.23):

Let (X, d) be a metric space and $\langle x_n \rangle$ be a sequence in X, we say that $\langle x_n \rangle$ is a Cauchy sequence if $\forall \epsilon > 0, \exists\ k = k(\epsilon)$ such that:

$$d(x_n, x_m) < \epsilon \qquad \forall\ n, m > k$$

Proposition (4.24):

Every convergence sequence in a metric space X is a Cauchy sequence.

Proof: Let $\langle x_n \rangle$ be a convergence sequence that converge to x_0 i.e $x_n \to x_0$.

Let $\epsilon > 0$, $\because$ $x_n \to x_0$, then $\exists\, k = k(\frac{\epsilon}{2})$ such that $d(x_n, x_0)$

$$d(x_n, x_m) \leq d(x_n, x_0) + d(x_m, x_0)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \forall\, n > k , \forall\, m > k$$

$$< \epsilon \qquad \forall\, n, m > k$$

Remark (4.25):

The converse of proposition (4.24) in general is not true.

Proof: Let $X = R - \{0\}$, $d(x, y) = |x - y| \quad \forall\, x, y \in R - \{0\}$

$\exists\, \langle \frac{1}{n} \rangle$ in $R - \{0\}$.

$\frac{1}{n} \to 0 \notin R - \{0\}$

$\therefore\ \langle \frac{1}{n} \rangle$ is not a convergence sequence

By proposition (4.24) is a Cauchy sequence but not converges in $R - \{0\}$.

Definition (4.26):

A metric space (X, d) is called a complete metric space if every Cauchy sequence in X is a convergence sequence in X.

Theorem (4.27):

R^k is called a complete metric space $\forall\, k \geq 1$.

Proof: $k = 2$ let $\langle (x_n, y_n) \rangle$ be a Cauchy sequence in R^2.

$\forall \epsilon > 0$, $\exists\ k_1 = k_1(\frac{\epsilon}{2})$ such that

$$d((x_n, y_n), (x_m, y_m)) = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} < \frac{\epsilon}{2} \quad \forall n, m > k_1$$

$$= (x_n - x_m)^2 + (y_n - y_m)^2 < \frac{\epsilon^2}{4} \quad \forall\ n, m > k_1$$

$$\therefore\ (x_n - x_m)^2 < \frac{\epsilon^2}{4} \qquad \forall\ n, m > k_1 \qquad \cdots\ (1)$$

$$\text{And}\ \ (y_n - y_m)^2 < \frac{\epsilon^2}{4} \qquad \forall\ n, m > k_1 \qquad \cdots\ (2)$$

$$|x_n - x_m| < \frac{\epsilon}{2} \qquad \forall\ n, m > k_1 \qquad \cdots\ (3)$$

$$\text{And}\ \ |y_n - y_m| < \frac{\epsilon}{2} \qquad \forall\ n, m > k_1 \qquad \cdots\ (4)$$

$\therefore$ $\langle x_n \rangle$ is a Cauchy sequence in R and $\langle y_n \rangle$ is a Cauchy sequence in R.

$\because$ R is complete

$\therefore$ $x_n \to x_0 \in R$ and $y_n \to y_0 \in R$

$\exists\ k_2 = k_2(\frac{\epsilon}{2})$ such that $|x_n - x_0| < \frac{\epsilon}{2}$ $\qquad \forall\ n > k_2$

$\exists\ k_3 = k_3(\frac{\epsilon}{2})$ such that $|y_n - y_0| < \frac{\epsilon}{2}$ $\qquad \forall\ n > k_3$.

Claim: $(x_n, y_n) \to (x_0, y_0) \in R^2$.

$$\Big(d((x_n, y_n), (x_0, y_0))\Big)^2 = (x_n - x_0)^2 + (y_n - y_0)^2$$

$$< \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{2} \qquad \forall n > k = \max\{k_1, k_2\}$$

H.W: In R^3