that p is a cluster point for S, if every open set contain p contains another element q in S and $p \neq q$

i.e for any open set U ; $p \in U \quad (U - \{p\}) \cap S \neq \emptyset$

Note:-

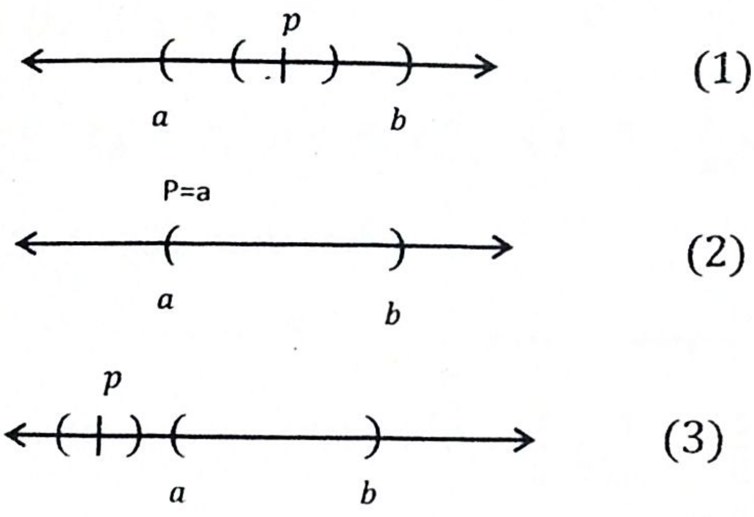
We will denote the set of all cluster points of S by $l(S)$.

($\bar{S} = S \cup l(S)$ is called the closer of S)

$l(S) = \{p : p \text{ is a cluster points of } S\}$

Example:-

1) $S = (a, b)$, $X = R$, find $l(S)$



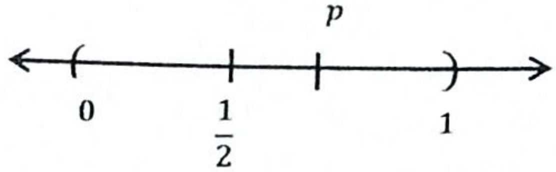
$\therefore\ l(S) = [a, b] \quad \Rightarrow \quad \bar{S} = [a, b]$

a) If $p \in S$, then any open interval U, $\exists\, p \in U$ we have:-$U - \{p\} \cap S \neq \emptyset$.
b) If $p = a$, then any open interval U contain $a = p$ satisfies :- $U \cap S \neq \emptyset$
c) For any $p \in R - [a, b]$, $p \neq a$, then $\exists\, U = (p - 4d, p + d)$, $U \cap S = \emptyset$ and $d = |a - p|$

2) Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\right\} \subseteq R$

a) If $p = \frac{1}{n}$, $n \in N$, take $U = \left(\frac{1}{n+1}, \frac{1}{n-1}\right)$, $\frac{1}{n} \in U$, $\left(U - \left\{\frac{1}{n}\right\}\right) \cap S = \emptyset$

b) If $p \neq \frac{1}{n}$, $p \neq \{0\}$, if $p > 0$, $\exists\, U = \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $\frac{1}{n} \in U$,



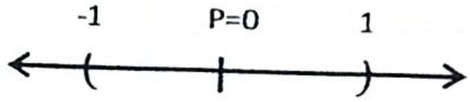
if $p > \frac{1}{n}$, $\exists\, U = \left(\frac{1}{n}, \frac{1}{n-1}\right)$ and $U \cap S = \emptyset$

c) If $p < 0$, $\exists\; U = (-\infty, 0)$, $p \in (-\infty, 0)$ and $U \cap S = \emptyset$

d) If $p = \{0\}$, then any open set (interval) contain 0, $0 \in \left(-\epsilon, \frac{\epsilon}{2}\right)$ and $U \cap S \neq \emptyset$.

Since $\forall\, \epsilon_2 > 0$, $\exists\, k \in Z^+$ $s.t$ $0 < \frac{1}{k} < \epsilon_2$, $\frac{1}{k} \in \left(-\epsilon, \frac{\epsilon}{2}\right)$

$\therefore\ l(S) = \{0\}$ only zero



$\bar{S} = S \cup l(S) = S \cup \{0\} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\right\}$

$l(S) = \{0\} \nsubseteq S$

$\therefore$ S not closed.

3) Let (X, d) be a metric space and S be any finite subset of X, then $l(S) = \emptyset$.

<u>Sol:</u> let $S = \{x_1, x_2, \cdots, x_n\} \subseteq X$, let $p \in X$, if $p \in S$, then $\exists\, t \in N$, $1 \leq t \leq n$ $s.t$ $p = x_t$.

Then $d(x_i, x_t) = d_i \quad \forall\, i = 1,2,\cdots,n \quad , \quad i \neq t$

$\epsilon = \min\{d_i \; : \; i = 1,2,\cdots,n \quad , \quad i \neq t\}$

$B_\epsilon(x_t) - \{x_t\} \cap S = \emptyset$

Now, $p \notin S \;\;,\;\; p \in X - S$

$p \neq x_i \quad \forall\, i = 1,2,\cdots,n$

$\therefore d(p, x_i) = r_i \quad \forall\, i = 1,2,\cdots,n$

Let $\epsilon < \min\{r_1, r_2, \cdots, r_n\} \quad \therefore B_\epsilon(p) \cap S = \emptyset$,

$\therefore p$ is not a cluster point

$\therefore l(S) = \emptyset \qquad and \quad \bar{S} = S \cup \emptyset = S$.

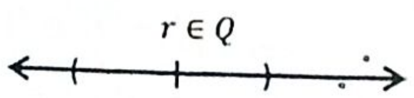
4) Let Q be the set of rational numbers in R with the usual distance.

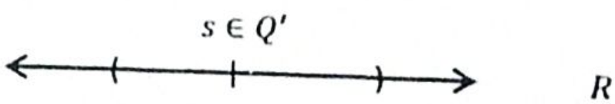
a) If $p \in Q$, then any open set (open interval) $U,\ s.t\ p \in U$ we have:- $(U - \{p\}) \cap Q \neq \emptyset$. (By the density rational number)

b) If $p \notin Q \;\rightarrow p \in Q'$, then any open set (open interval) U such that $p \in U$, we have $U \cap Q \neq \emptyset$. (By the density irrational number)

$\therefore l(Q) = R \qquad and \quad \bar{Q} = Q \cup l(Q)$

$\qquad\qquad\qquad\qquad\qquad = Q \cup R$.





<u>(H.W)</u>

Find $l(Q')$, $l(Z)$; $Z \subseteq R$.