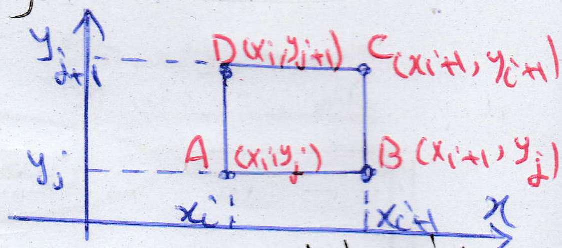
Double Integration



Suppose we have

$$I = \int_a^b \int_c^d f(x, y)\, dx\, dy$$

The region of integration for which is the area bounded by the lines

$$x = c,\ x = d\ ;\ y = a,\ y = b$$

in the xy-plane. consider, the double integral

$$I = \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} f(x, y)\, dx\, dy \quad \text{———} \quad (1)$$

where $x_{i+1} = x_i + h$, and, $y_{j+1} = y_j + K$, and,
$x_{i+1} - x_i = h$, $y_{j+1} - y_j = K$ for all i, j.

Now apply trapezoidal rule repeated to get the value of I in (1), we have

$$I = \int_{y_j}^{y_{j+1}} \frac{h}{2} \left[f(x_i, y) + f(x_{i+1}, y) \right] dy$$

$$= \frac{h}{2} \left[\int_{y_j}^{y_{j+1}} f(x_i, y)\, dy + \int_{y_j}^{y_{j+1}} f(x_{i+1}, y)\, dy \right]$$

$$= \frac{h}{2} \left[\frac{K}{2} \left\{ f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1}) \right\} \right]$$

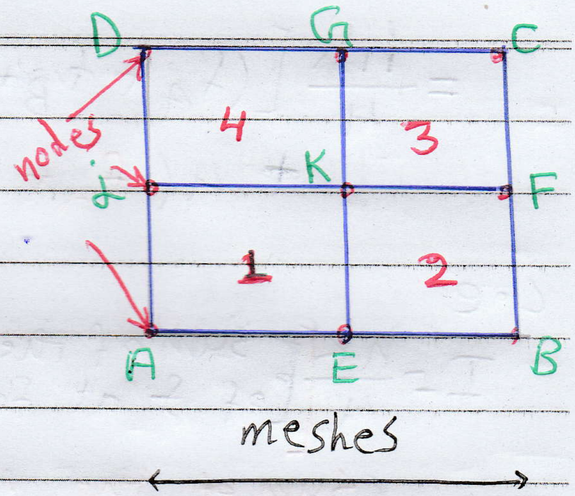
$$= \frac{hK}{4} \left[f_{ij} + f_{i,j+1} + f_{i+1,j} + f_{i+1,j+1} \right] \quad \text{———} \quad (2)$$

By using the Figure, we get

$$I = \frac{hK}{4} \left[\text{Sum of the values of } f(x, y) \text{ at four corner} \right]$$

$$= \frac{hK}{4} \left[f_A + f_D + f_B + f_C \right] \quad \text{———} \quad (3)$$

Divide ABCD into meshes by dividing AB into 2-parts each length h and AD into 2-equal parts with length K



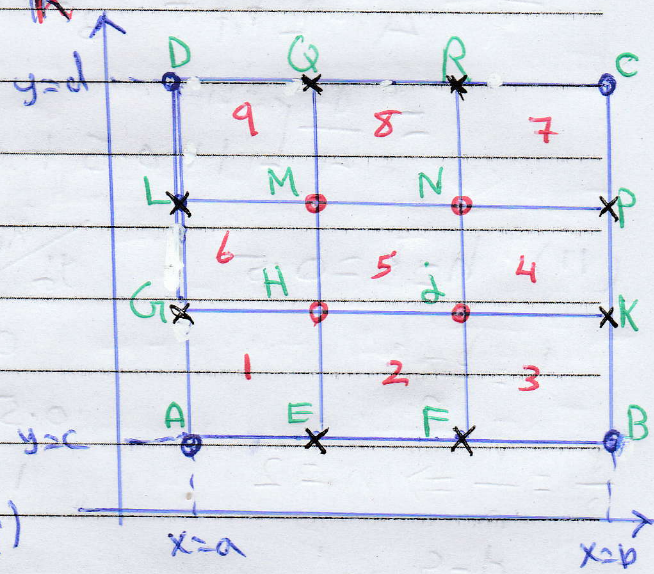
$$I = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \iint_{AEKj} + \iint_{EBFK} + \iint_{KFCG} + \iint_{jKGD}$$

$$= \frac{hk}{4}\left[(f_A + f_E + f_K + f_j) + (f_E + f_B + f_F + f_K) + (f_K + f_F + f_C + f_G) \right.$$
$$\left. + (f_K + f_G + f_D + f_j) \right]$$

$$= \frac{hk}{4}\left[f_A + f_B + f_C + f_D + 2(f_E + f_G + f_F) + 4 f_K \right]$$

again if Divide ABCD into meshes by dividing AB into 3-parts each length h and AD into 3-equal parts with length K



$$I = \int_c^d \int_a^b f(x,y)\,dx\,dy = \iint_{I_1} + \iint_{I_2} + \iint_{I_3}$$
$$+ \iint_{I_4} + \iint_{I_5} + \cdots + \iint_{I_9}$$

$$= \frac{hk}{4}\left[(f_A + f_E + f_H + f_G) \right.$$
$$+ (f_E + f_F + f_j + f_H) + (f_F + f_B + f_K + f_j)$$
$$+ (f_j + f_K + f_P + f_N) + (f_H + f_j + f_N + f_M) + (f_G + f_H + f_M + f_L)$$
$$\left. + \cdots + (f_L + f_M + f_Q + f_D) \right]$$

$$= \frac{hk}{4}\Big[(f_A + f_B + f_C + f_D) + 2(f_E + f_F + f_K + f_P + f_R + f_Q + f_L + f_G)$$
$$+ 4(f_H + f_J + f_M + f_N)\Big]$$

i.e

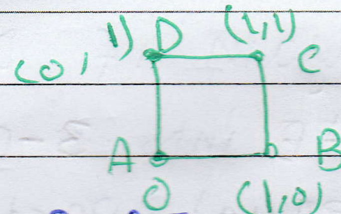
$$I = \frac{hk}{4}\Big[\begin{Bmatrix}\text{sum of the values} \\ \text{of } f \text{ at four corners}\end{Bmatrix} + 2\begin{Bmatrix}\text{sum of the values of } f \text{ at} \\ \text{remaining nodes on the boundary}\end{Bmatrix}$$
$$+ 4\begin{Bmatrix}\text{sum of the values of } f \text{ at} \\ \text{the interior nodes}\end{Bmatrix}\Big]$$

This equation is known as Trapezoidal rule four double integration.

Ex: Calculate $\int_0^1 \int_0^1 \frac{dx\,dy}{1+x+y}$ $h = k = 0.5$, $h = k = 1$; $h = k = 0.25$

(i) $h = k = 1$

$y_j \backslash x_i$	0	1
0	1	0.5
1	0.5	0.33



$$I = \frac{hk}{4}[f_A + f_B + f_C + f_D] = \frac{hk}{4}[f_{00} + f_{10} + f_{11} + f_{01}]$$

$$= \frac{1}{4}[1 + 0.5 + 0.33 + 0.5] = \frac{2.33}{4} = 0.5825$$

(ii) $h = k = 0.5$

$h = \frac{b-a}{n}$

$\frac{1}{2} = \frac{1}{n} \Rightarrow n = 2$

$y_j \backslash x_i$	0	0.5	1
0	1	0.66 *	0.50
0.5	0.66 *	0.50	0.40 *
1	0.50	0.40 *	0.33

$k = \frac{d-c}{m} \Rightarrow \frac{1}{2} = \frac{1}{m} \Rightarrow m = 2$

$$I = \frac{(0.5)(0.5)}{4}[1 + 0.50 + 0.50 + 0.33] + 2(0.66 + 0.40 + 0.40 + 0.66) + 4(0.5)] = 0.5356.$$

Ex: Using Trapezoidal rule to calculate

$$I = \int_1^2\int_1^2 \frac{dx\,dy}{x+y}$$ ~~taking for subintervals,~~ $n = 1, 2, 4$

Soln: $n=4$ $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} = 0.25$; $k = \frac{d-c}{m} = \frac{2-1}{4} = 0.25$

$x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.50$, $x_3 = 1.75$, $x_4 = 2$

$y_0 = 1$, $y_1 = 1.25$, $y_2 = 1.50$, $y_3 = 1.75$, $y_4 = 2$

$$f(x,y) = \frac{1}{x+y}$$

$m = n = 1$; $h = \frac{b-a}{n} \Rightarrow h = 1$, $k = 1$

$$I_t = \frac{hk}{4}\left[f_{00} + f_{n0} + f_{0m} + f_{nm}\right] = \frac{hk}{4}\left[f_A + f_B + f_C + f_D\right]$$

$$= \frac{1}{4}[0.5 + 0.3 + 0.3 + 0.25] = \frac{1.35}{4} = 0.3375$$

y \ x	1	2
1	0.50	0.30
2	0.30	0.25

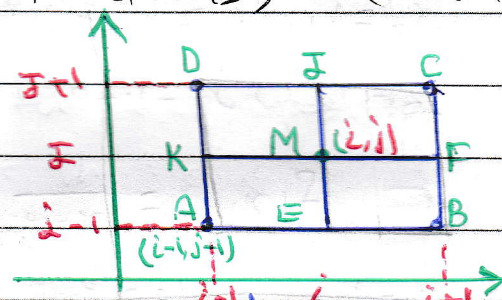
$h = k = 0.25$, $n = m = 4$

y \ x	1	1.25	1.50	1.75	2
1	0.5	0.44			0.33
1.25	0.44	0.40			0.36
1.50	0.40	0.36			0.28
1.75	0.36	0.33			0.26
2	0.33	0.30			0.25

$$I = \frac{1}{64}\left[f_{1,1} + f_{1,2} + 2(f_{1,1.25} + f_{1,1.5} + f_{1,.75}) + (f_{2,1} + f_{2,2}) + 2(f_{2,1.25} + f_{2,1.5} + f_{2,1.75}) + 2(f_{1.25,1} + f_{1.25,2}) + 2(f_{1.25,1.25} + f_{1.25,1.5} + f_{1.25,1.75})\right) + \cdots = 0.3407$$

Simpson's Rule

Divide the interval (a,b) into $2n$ equal subintervals each of length h, and, the interval (c,d) into $2m$ equal subintervals each of length K. Then we applying Simpson's rule in both directions, we have



$$I_S = \int_{y_{j-1}}^{y_{j+1}} \left(\int_{x_{i-1}}^{x_{i+1}} f(x,y)\,dx \right) dy$$

$$= \frac{h}{3} \int_{y_{j-1}}^{y_{j+1}} \left[\underbrace{f(x_{i-1},y)}_{\mathrm{I}} + 4\,\underbrace{f(x_i,y)}_{\mathrm{II}} + \underbrace{f(x_{i+1},y)}_{\mathrm{III}} \right] dy$$

$$= \frac{hK}{9} \Big[\underbrace{\big(f(x_{i-1},y_{j-1}) + 4 f(x_{i-1},y_j) + f(x_{i-1},y_{j+1})\big)}_{\mathrm{I}} +$$

$$+ 4\underbrace{\big(f(x_i,y_{j-1}) + 4 f(x_i,y_j) + f(x_i,y_{j+1})\big)}_{\mathrm{II}} +$$

$$\underbrace{\big(f(x_{i+1},y_{j-1}) + 4 f(x_{i+1},y_j) + f(x_{i+1},y_{j+1})\big)}_{\mathrm{III}} \Big]$$

$$= \frac{hK}{9} \Big[f_{i-1,j-1} + 4 f_{i-1,j} + f_{i-1,j+1}) + 4(f_{i,j-1} + 4 f_{ij} + f_{i,j+1})$$

$$+ (f_{i+1,j-1} + 4 f_{i+1,j} + f_{i+1,j+1} \Big]$$

$$= \frac{hK}{9} \left[(f_A + f_B + f_C + f_D) + 4(f_E + f_F + f_J + f_K) + 16 f_M \right. \quad (2)$$

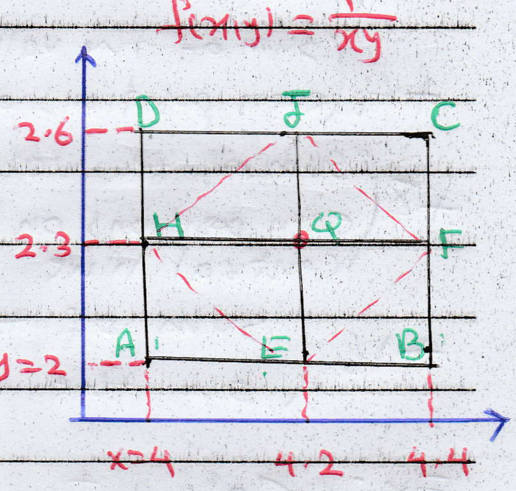
$$I_S = \frac{hK}{9} \left[\left(\begin{array}{c} \text{sum of the values} \\ \text{of } f \text{ at 4-corners} \end{array} \right) + 4 \left(\begin{array}{c} \text{sum of the values of} \\ f \text{ at the remaining nodes} \\ \text{in the boundary} \end{array} \right) + 16 \left(\begin{array}{c} \text{value of } f \\ \text{at the} \\ \text{central point} \end{array} \right) \right]$$

Ex: use Simpson's rule to calculate the integral

$$I = \int_{y=2}^{2.6} \int_{x=4}^{4.4} \frac{dx\,dy}{xy}, \quad n = m = 2$$

$\underline{\text{Sol}^n}$: $h = \frac{b-a}{2n} = \frac{b-a}{2} = \frac{4.4-4}{2} = \frac{0.4}{2} = 0.2$

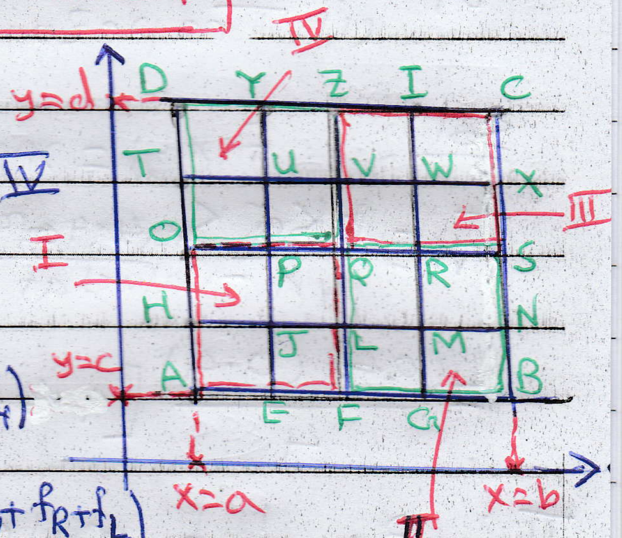
$K = \frac{d-c}{2m} = \frac{2.6-2}{2} = \frac{0.6}{2} = 0.3$



$$I_S = \frac{hK}{9}\Big[\big(f(4,2) + f(4.4,2) + f(4.4,2.6) + f(4,2.6)\big) + 4\big(f(4.2,2) + f(4.4,2.3) + f(4.2,2.6) + f(4,2.3)\big) + 16\, f(4.2,2.3)\Big] = 0.025$$

$\underline{\text{EX}}$: Let $I_S = \int_c^d \int_a^b f(x,y)\,dx\,dy$

$h = \frac{(b-a)}{4}$, $K = \frac{(d-c)}{4}$ $\boxed{n = m = 4}$



We apply Simpson's rule ② to 4-rectangles I, II, III and IV ((AFQO, FBSQ, QSCZ, OQZD)) I

$$I = \frac{hK}{9}\Big[(f_A + f_F + f_Q + f_O) + 4(f_E + f_L + f_P + f_H) + 16 f_J) + (f_F + f_B + f_S + f_Q) + 4(f_G + f_N + f_R + f_L) + 16 f_M) + ((f_Q + f_S + f_C + f_Z) + 4(f_R + f_X + f_I + f_V) + 16 f_W) + ((f_Q + f_Z + f_D + f_O) + 4(f_P + f_V + f_Y + f_I) + 16 f_U)\Big]$$

$$= \frac{hK}{9}\Big[(f_A + f_B + f_C + f_D) + 2(f_F + f_S + f_Z + f_O) + 4(f_E + f_G + f_N + f_X + f_I + f_Y + f_T + f_H) + 4 f_Q + 8(f_P + f_R) + 8(f_L + f_V) + 16(f_J + f_M + f_W + f_U)\Big]$$

$$= \frac{hk}{9}\left[\begin{pmatrix}\text{sum of the values}\\ \text{of } f \text{ at 4-corners}\end{pmatrix} + 2\begin{pmatrix}\text{sum of the values of } f \text{ at}\\ \text{odd positions on the boundary}\end{pmatrix}\right.$$

$$+4\begin{pmatrix}\text{sum of the values } f \text{ at}\\ \text{even positions on the boundary}\end{pmatrix} + 4\begin{pmatrix}\text{the value of } f \text{ at}\\ \text{the centre point}\end{pmatrix}$$

$$+8\begin{pmatrix}\text{sum of the values of } f \text{ at}\\ \text{even positions on horizontal}\\ \text{central line}\end{pmatrix} + 8\begin{pmatrix}\text{sum of the values of } f \text{ at}\\ \text{even positions on vertical}\\ \text{central line}\end{pmatrix}$$

$$\left. +16\begin{pmatrix}\text{sum of the values of } f \text{ at the even positions}\\ \text{on the even rows of the matrix}\end{pmatrix}\right]$$

Ex: Calculate $\displaystyle\int_{2}^{2.4}\int_{4}^{4.4} xy\,dx\,dy$

by (i) Trapezoidal rule (II) Simpson's $\frac{1}{3}$ rule.

Sol^n: $f(x,y) = xy$, $h = 4.4 - 4 = 0.4$, $k = 2.4 - 2 = 0.4$

y \ x	4	4.4
2	8	8.8
2.4	9.6	10.56

$I_t = \frac{hk}{4}\left[\begin{matrix}\text{Sum of the}\\ \text{values at 4-corners}\end{matrix}\right]$

$= \frac{(0.4)(0.4)}{4}[9.6 + 10.56 + 8.8 + 8] = 1.4784$

II: By $\frac{1}{3}$ h rule:

y \ x	4	4.2	4.4
2	8	8.4	8.8
2.2	8.8	9.24	9.68
2.4	9.6	10.08	10.56

$h = \frac{b-a}{2} = \frac{4.4-4}{2} = \frac{0.4}{2} = 0.2$

$k = \frac{d-c}{2} = \frac{2.4-2}{2} = 0.2$

$I_s = \frac{\quad}{9}[(9.6 + 10.56 + 8.8 + 8) + 4(10.08 + 9.68 + 8.4 + 8.8)$
$+ 16(9.24)] = 1.33056$

Check: Compare the value with exact value by actual integration, $I_{exact} = \iint xy\,dA = 1.4784$

Exercises 1

1- Apply Trap. rule to evaluate

$$\int_{1}^{5}\int_{1}^{5} \frac{dx\,dy}{\sqrt{x^2+y^2}}$$, taking 2-subintervals.

2- Using Simpson's rule, evaluate

$$\int_{1}^{2.8}\int_{2}^{3.2} \frac{dx\,dy}{x+y} \quad ; \quad \int_{0}^{1}\int_{0}^{1} e^{x+y}\,dx\,dy$$

3- $$\int_{y=0.2}^{0.6}\int_{x=1.5}^{3} xy\,dx\,dy \quad \text{if} \quad h=0.5,\ k=0.1$$

By using Trap. rule and Simpson's rule then Compare with actual error.

4- Calculate the given integral by Trap. rule for $n=2, 4, 8, 16, 32$

$$\int_{0}^{10} \frac{300x}{1+e^x}\,dx \quad , \quad \int_{0}^{2} \frac{1}{\sqrt{x}}\,dx$$

5- Compute $$I=\int_{a=8}^{b=30}\left[2000\ln\left(\frac{140{,}000}{140{,}000-2100x}\right)-9.8x\right]dx$$

using Simpson 1/3 rule ($n_1=4$), and Simpson 3/8 rule (with $n_2=3$)

ملاحظة:

اذا ما كانت عدد التقسيمات n (على محور x) غير مساوية الى عدد التقسيمات m (على محور y)

فهل بالإمكان حل السؤال بأي من الطريقتين ؟ . اعط مثال على اجابتك .