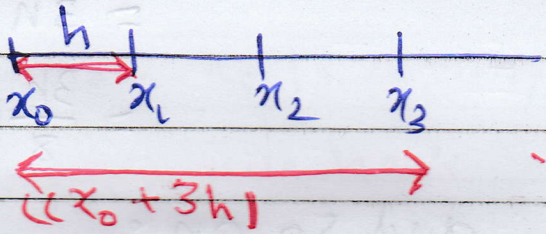
"ثانياً":

Simpson's three-eighth rule ($\frac{3}{8}$ rule)

$$I = \frac{3}{8} h \left[f(x_0) + 3 f_1 + 3 f_2 + f(x_3) \right]$$

طريقة $\frac{3}{8}h$ البسيطة حيث $n=3$

$$I = \int_{x_0}^{x_0+3h} f(x)\,dx$$

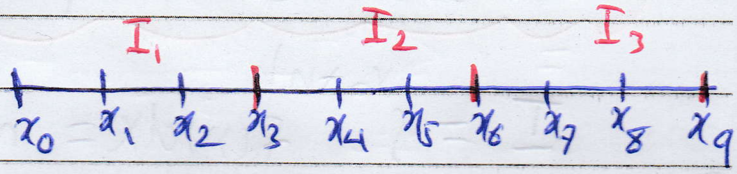


وللحصول على الطريقة المركبة يجب ان تكون عدد التقسيمات من مضاعفات 3 اي ان

$$I = \int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} + \cdots$$

ونطبق الصيغة البسيطة على كل جزء من الشكل لنحصل على الصيغة التالية

$$I = \frac{3}{8} h \left[f(x_0) + f(x_n) + 3\{ f_1 + f_2 + f_4 + f_5 + \cdots + f_{n-1} \} + 2\{ f_3 + f_6 + \cdots + f_{n-3} \} \right]$$



ولغرض الحصول على الصيغ اعلاه يمكن ان نستخدم صيغ "نيوتن - كوتس" حيث $n=3$ وان المنحني يمر بالنقاط (x_i, y_i) حيث $(i = 0,1,2,3)$ "الصيغ البسيطة" والحصول على متعددة حدود تكعيبية والتي تقرب الدالة $y=f(x)$ فحسب الشكل على اي ان:

$$\int_{x_0}^{x_0+3h} f(x)\,dx \approx \int_{x_0}^{x_0+3h} P_3(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{2}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0 \right) = \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3 \right)$$

Similarly:

$n=3$

$x_3 \quad x_4 \quad x_5 \quad x_6$

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \int_{x_3}^{x_6} f(x)\,dx$$

$$= 3h\left(y_3 + \frac{3}{2}\Delta y_3 + \frac{3}{2}\Delta^2 y_3 + \frac{1}{8}\Delta^3 y_3\right)$$

$$= \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

and so on:

Adding all such from x_0 to x_0+nh, where n is a multiple of 3, we get

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\Big[(y_0+y_n) + 3(y_1+y_2+y_4+y_5 + \cdots y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\Big]$$

ملاحظة ⟸ يجب ان تكون الفترات الجزئية من مضاعفات العدد 3 .

Newton - Cotes Quadrature Formula:

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = nh\Big[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\Big], \quad n=1,2,3,\ldots \qquad (*)$$

Boole's Rule

رائع!

putting $n=4$ in $(*)$

$$I = \int_{x_0}^{x_0+4h} f(x)\,dx = 4h\left(y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right) = \frac{2h}{45}\left(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right)$$

Similarly:

$$I_2 = \int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}\left[7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8\right]$$

and so on

Adding all these integrals from x_0 to x_0+nh, where n is a multiple of 4, we have

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \cdots\right]$$

This rule is known as Boole's Rule.

* عدد التقسيمات (الفترات الجزئية n) يجب أن يكون من مضاعفات 4.

Ex: Calculate $\int_0^6 \frac{dx}{1+x^2}$ by using

(i) Trapezoidal rule
(ii) Simpson's $\frac{1}{3}$ rule (iii) Simpson's $\frac{3}{8}$ rule

Soln: $I = [0,6]$, Divide into 6 parts each of width $h=1$. we get

x_i	0	1	2	3	4	5	6
$f(x_i)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(iii) By 3/8 rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right]$$

$$= \frac{3}{8}\left[(1+0.027) + 3(0.5+0.2+0.0588+0.0385) + 2(0.1)\right] = 1.3571$$

$$I_{exact} = \int_0^6 \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^6 = \tan^{-1} 6 = 1.4056$$

$$\text{actual error} = |I_{exact} - I_{Appr.}| = |1.4056 - 1.3571|$$

$$= 0.0485$$

Ex: The velocity v (Km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows:

t	2	4	6	8	10	12	14	16	18	20
v	10	18	25	29	32	20	11	5	2	0

Estimate((approximate)) the distance covered in 20 mins

Soln:
$$v = \frac{ds}{dt} \Rightarrow ds = v\,dt$$

$$\int ds = \int v\,dt \Rightarrow S\Big|_{t=0}^{20} = \int_{t=0}^{t=20} v\,dt$$

Hence the required distance

$$= S\Big|_{t=0}^{t=20} = \frac{h}{3}\left[v_0 + v_{10} + 4\{v_1 + v_3 + v_5 + v_7 + v_9\} + 2\{v_2 + v_4 + v_6 + v_8\}\right]$$

$$= \frac{2}{3}[0 + 4(80) + 2(72)] = 309.33 \text{ Km}.$$

Error in Simpson's $\frac{3}{8}$ h

$$I = [x_0, x_3] \quad , \quad \epsilon = -\frac{3h^5}{80} y^{(IV)}$$

5: Forward Difference Integration Formulae ملاحظة

(i) ~~Newton Forward formula (Gregory-Newton)~~

$$f_p = f_0 + p\Delta f_0 + \frac{1}{2}p(p-1)\Delta^2 f_0 + \dots + \binom{p}{n}\Delta^n f_0 + \dots$$

s.t. $f_0 = f(x_0) = y_0$; $f_p = f(x)$

$x = x_0 + ph$,

The function being tabulated ~~at~~ equal intervals h. In this formula we have the f^n $f(x)$ expressed in terms of the variable p which satisfies the linear relation $x = x_0 + ph$ in which x_0 and h are constants

$$x = x_0 + ph \Rightarrow \frac{dx}{dp} = h \Longrightarrow dx = h\,dp$$

$$\int f(x)\,dx = \int f(x_0+ph)\,dx = \int f_p\, h\, dp$$

$$= h \int f_p\, dp$$

$$\int_{x_0}^{x_n} f(x)\,dx = h\int_0^n f_p\,dp \quad (h \text{ is a constant})$$

$$= h\int_0^n \left(f_0 + p\Delta f_0 + \frac{p^2-p}{2}\Delta^2 f_0 + \frac{p^3-3p^2+2p}{6}\Delta^3 f_0 + \dots\right)dp$$

$$= h\left[p f_0 + \frac{1}{2}p^2\Delta f_0 + \left(\frac{p^3}{6} - \frac{p^2}{4}\right)\Delta^2 f_0 + \left(\frac{p^4}{24} - \frac{p^3}{6} + \frac{p^2}{6}\right)\Delta^3 f_0 + \dots\right]_0^n$$

$$= h\left[n f_0 + \frac{n^2}{2}\Delta f_0 + \frac{n^2(2n-3)}{12}\Delta^2 f_0 + \frac{n^2(n-2)^2}{24}\Delta^3 f_0 + \dots\right]$$

putting $n=1$ (one-interval), we have

$$\int_{x_0}^{x_1} f(x)\,dx = h\left[f_0 + \frac{1}{2}\Delta f_0 - \frac{1}{12}\Delta^2 f_0 + \frac{1}{24}\Delta^3 f_0 + \dots\right]$$

If $n=2$ (two intervals), we have

$$\int_{x_0}^{x_2} f(x)dx = h\left[2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0 - \frac{1}{90}\Delta^4 f_0 + \cdots\right]$$

$$f_0 = y_0, \quad \Delta f_0 = f_1 - f_0 = y_1 - y_0; \quad \Delta^2 f_0 = \Delta f_1 - \Delta f_0$$
$$= y_2 - 2y_1 + y_0$$

Hence

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2 - \frac{1}{30}\Delta^4 f_0 + \cdots\right]$$

error

$n=2$, $\Delta^3 f \to 0$

$$\simeq \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$ Simpson's $\frac{1}{3}$ h rule.

Ex: Find $\int_0^{\pi/2} \sin x \, dx$ by tabulating $\sin x$ for $x = 0^\circ (15^\circ) 120^\circ$

i	x_i	$\sin x$	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$	$\Delta^4 f_0$
0	0°	0				
			0.2588			
1	15°	0.2588		−0.176		
			.2412		−0.165	0.24
2	30°	0.5000		−.341		
			.2071		−.141	
3	45°	0.7071		−.482		0.33
			.1589		−.108	
4	60°	0.8660		.590		0.40
			.0999		−.68	
5	75°	0.9659		.658		0.44
			.341		−.24	
6	90°	1.0		.682		0.48
			−.341		0.24	
7	105°	0.9659		−0.658		
			−.999			
8	120°	0.8660				

$$h = \frac{\pi}{12} = 0.26180$$

$$\int_0^{\pi/2} \sin x \, dx \simeq h\left[2(f_0 + f_2 + f_4) + 2(\Delta f_0 + \Delta f_2 + \Delta f_4) + \frac{1}{3}(\Delta^2 f_0 + \cdots) - \frac{1}{90}(\Delta^4 f_0 + \cdots)\right]$$

$$\cong h\left[2(0+0.5000+0.8660)+2(0.2588+0.2071+0.0999)\right.$$
$$\left.+\frac{1}{3}(-0.176-0.482-0.658)-\frac{1}{90}(0.24+0.40+0.48)\right)$$

$$= h[2.7320+1.1316+(-0.04387)-0.00012]$$
$$= h(3.8636-0.0440) = 0.26180\,(3.8196)$$
$$= 0.99997\ldots$$
$$\simeq 1.$$

6) Central Difference Integration Formula

(i) Using Bessel's Interpolation formula

$$f_p = f_0 + p\,\delta f_{0.5} + B_2(\delta^2 f_0 + \delta^2 f_1) + B_3\,\delta^3 f_{0.5}$$
$$+ B_4(\delta^4 f_0 + \delta^4 f_1) + \cdots$$

where $B_2 = \dfrac{p(p-1)}{4}$, $B_3 = \dfrac{p(p-1)(p-\frac{1}{2})}{6}$

$$B_4 = \frac{(p+1)(p)(p-1)(p-2)}{48}$$

$$\int_0^p B_2\,dp = \left.\frac{p^3}{12}-\frac{p^2}{8}\right]_0^p = \frac{p^2(2p-3)}{24}$$

$$\int_0^p B_3\,dp = \frac{1}{6}\left[\frac{p^4}{4}-\frac{p^3}{2}+\frac{p^2}{4}\right]_0^p = \frac{p^2(p-1)^2}{24}$$

$$\int_0^p B_4\,dp = \frac{1}{48}\left[\frac{p^5}{5}-\frac{p^4}{2}-\frac{p^3}{3}+p^2\right] = \frac{p^2(6p^3-15p^2-10p+30)}{30\times 48}$$

$$\int_{x_0}^{x} f(x)\,dx = h\int_0^1 \left[f_0 + p\,\delta f_{\frac{1}{2}} + B_2(\delta^2 f_0 + \delta^2 f_1) + \cdots\right] dp$$
$$= h\left[f_0 + \frac{1}{2}\delta f_{\frac{1}{2}} - \frac{1}{24}(\delta^2 f_0 + \delta f_1) + \cdots\right]$$
$$= h\left[\frac{f_0+f_1}{2} - \frac{\delta^2 f_0 + \delta^2 f_1}{24} + \cdots\right]$$