Initial-value Problems for ordinary differential equations

Def.:- A function $f(t,y)$ is said to satisfy a Lipschitz condition on the variable y on a set $D \subset R^2$, if a constant $L > 0$, exists with

$$|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$$

Whenever (t,y_1) and (t,y_2) are in D. The constant L is called Lipschitz constant for f

ex:- Show that $f(t,y) = t|y|$ satisfies a Lipschitz condition in $D = \{(t,y) \mid 1 \le t \le 2 \text{ and } -3 \le y \le 4\}$

Sol for each pair of Points (t,y_1) and (t,y_2) in D, we have

$$|f(t,y_1) - f(t,y_2)| = \big|t|y_1| - t|y_2|\big| \le |t| \cdot |y_1 - y_2|$$

$$\le 2|y_1 - y_2|$$

$L > 0 \Rightarrow$ then the function is satisfy Lipschitz Condition

Theorem:- Suppose that $D = \{(t,y) \mid a \leq t \leq b$ and ~~that~~ $-\infty < y < \infty\}$ and that $f(t,y)$ is a continuous on D. If f ~~is~~ satisfies a Lipschitz condition on D in the variable y then the initial-value problem

$$y'(t) = f(t,y) \qquad a \leq t \leq b \quad , \quad y(a) = \alpha$$

has a unique Solution $y(t)$ for $a \leq t \leq b$

Ex:- Show that there is a unique solution for IVP

$$y' = 1 + t\sin(ty) \quad , \quad 0 \leq t \leq 2 \quad , \quad y(0) = 0$$

Sol Holding t constant and Applying the Mean-val theorem to the function

$$f(t,y) = 1 + t\sin(ty)$$

we find that when $y_1 < y_2$ a number ξ in (y_1, y_2) exist with

$$\frac{f(t,y_2) - f(t,y_1)}{y_2 - y_1} = \frac{\partial}{\partial y} f(t,\xi) = t^2 \cos(\xi t)$$

Thus

$$|f(t,y_2) - f(t,y_1)| = |y_2 - y_1| \, |t^2 \cos(\xi t)| \leq 4 \, |y_2 - y_1|$$

and f satisfies a Lipschitz condition in the var
y with Lipschitz constant $L = 4$. Addionally, $f(t,y)$
continuous when $0 \le t \le 2$, and $-\infty \le y \le \infty$ So, theore
implies that a unique solution exists to this initia
value problem.

Theorem:- $^{(**)}$ Suppose $D = \{(t,y) \mid a \le t \le b$ and $-\infty < y < \infty\}$
If f is continuous and satisfies a Liptschitz
condition in the variable y on the set D, then th
IVP
$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha$$
is well-posed

Def.:- The initial value problem
$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha$$
is said to be a well-posed problem if

* A unique solution, $y(t)$, to the problem exists, and
* There is exist constant constant $\varepsilon_0 > 0$ and $k > 0$

Such that for any ε in $(0, \varepsilon_0)$ whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ $\forall$ $t \in [a,b]$ and $|\delta_0| < \epsilon$, the IVP

$$\frac{dz}{dt} = f(t,z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \alpha + \delta_0$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < K\varepsilon \quad \text{for all } t \in [a,b]$$

Ex.:- Show that IVP

$$\frac{dy}{dt} = y - t^2 + 1 \quad 0 \leq t \leq 2, \quad y(0) = 0.5 \quad \text{---} (1)$$

is well-posed on $D = \{(t,y) \mid 0 \leq t \leq 2$ and $-\infty < y <$

Sol

Because $\left| \frac{\partial(y - t^2 + 1)}{\partial y} \right| = |1| = 1$

Theorem(*) implies that $f(t,y) = y - t^2 + 1$ satisfies Lipschtiz condition in y on D with Lipschitz cons $L = 1$, since f is ~~constant~~ continuous on D. theore (**) implies that the problem is well-posed

$$\frac{dz}{dt} = z - t^2 + 1 + \delta, \quad 0 \leq t \leq 2, \quad z(0) = 0.5 + \delta_0 \quad \text{—(2)}$$

Where δ and δ_0 are constants. The solution to Eqs (1) and (2) are

$$y(t) = (t+1)^2 - 0.5e^t \quad \text{and} \quad \cancel{z(t) = (t+1)^2}$$

$$z(t) = (t+1)^2 + (\delta + \delta_0 - 0.5)e^t - \delta,$$

respectively

Suppose that ε is a positive number if $|\delta| < \varepsilon$ and $|\delta_0| < \varepsilon$, then

$$|y(t) - z(t)| = |(\delta + \delta_0)e^t - \delta| \leq |\delta + \delta_0|e^2 + |\delta|$$
$$\leq (2e^2 + 1)\varepsilon$$

for all t. This implies that, the problem is well-posed with $k(\varepsilon) = 2e^2 + 1$ for all $\varepsilon > 0$

Euler's method

Euler's method is the most elementray approximation technique for solving IVPs. Although it is seldom used in practice, the simplicity of ~~the~~ its derivation can be used to illistruate the techniques involved in the construction of some of the more advanced technique.

The object of Euler's method is obtained approximations to the well-posed IVP

$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha \quad \text{---} \quad (*)$$

We first make the stipulation that the mesh points are equally distrebuted throughout the interval $[a,b]$. This condition is ensured by choosing a positive integer N, setting $h = \frac{b-a}{N}$, and selecting the mesh points

$$t_i = a + ih, \quad \forall i = 0,1,2,\ldots,N$$

We will use Taylor's theorem to derive Euler's method, suppose that $y(t)$, the unique solution to (*) has two continuous derivatives on $[a,b]$, so that for each $i=0,1,2,\dots,N-1$

$$y(t_{i+1}) = y(t_i) + (t_{i+1}-t_i)\, y'(t_i) + \frac{(t_{i+1}-t_i)^2}{2}\, y''(\xi_i)$$

for some $\xi_i \in (t_i, t_{i+1})$, Because $h = t_{i+1}-t_i$, we have

$$y(t_{i+1}) = y(t_i) + h\, y'(t_i) + \frac{h^2}{2}\, y''(\xi_i)$$

and, because $y(t)$ satisfies differential eq. (*)

$$y(t_{i+1}) = y(t_i) + h\, f(t_i, y(t_i)) + \frac{h^2}{2}\, y''(\xi_i)$$

Euler's method constructs $w_i \approx y(t_i)$, $\forall\, i=1,2,\dots,N$ by deleting the remainder term. Thus Euler's method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + h f(t_i, w_i) \quad , \forall\, i=0,1,\dots,N-1$$

Ex:- Use the Euler's method to solve

$$y' = y - t^2 + 1 \quad , y(0) = 0.5 \qquad 0 \le t \le 2$$

Error bounds for ~~of~~ Euler's method

To derive an error bound for Euler's method -
we need to computitional lemmas.

Lemma ①:- For all $x \geq -1$ and any positive
m, we have $0 \leq (1+x)^m \leq e^{mx}$

Proof:- Applying Taylor's theorem with $f(x) = e^x$
$x_0 = 0$ and $n = 1$ gives

$$e^x = 1 + x + \frac{1}{2}x^2 e^{\xi} \quad , \quad 0 < \xi < x \text{ , thus}$$

$$0 < 1 + x + \frac{1}{2}x^2 e^{\xi} = e^x$$

and because $1 + x \geq 0$, we have

$$0 \leq (1+x)^m \leq (e^x)^m = e^{mx}$$

Theorem: Suppose that f is continuous and satisfies a Lipschitz condition with constant L on $D = \{(t,y) \mid a \le t \le b \text{ and } -\infty < y < \infty\}$ and that M exists with

$$|y''(t)| \le M \quad \forall\, t \in [a,b]$$

Where $y(t)$ denotes the unique solution to the initial-value problem

$$y'(t) = f(t,y) \qquad a \le t \le b \qquad y(a) = \alpha$$

let $w_0, w_1, \dots, w_N$ be the approximations generated by Euler's method for some positive integer N, then, for each $i = 0, 1, \dots, N$

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right]$$

Ex:- The solution to the IVP

$y' = y - t^2 + 1 \qquad 0 \leq t \leq 2 , \; y(0) = 0.5$

Was approximated with $h = 0.2$. find the error bound and compare with actual errors

Sol :-

Because $f(t,y) = y - t^2 + 1$, we have $\frac{\partial f(t,y)}{\partial y} = 1$ for all y, so $L = 1$, for this problem the exact Solution is $y(t) = (t+1)^2 - 0.5 e^t$, so $y''(t) = 2 - 0.5 e^t$

and $|y''(t)| \leq 0.5 e^2 - 2$

since $h = 0.2$, $L = 1$

$M = 0.5 e^2 - 2$ gives

$$|y_i - w_i| \leq 0.1 (0.5 e^2 - 2)(e^{t_i} - 1)$$

Hence

$$|y(0.2) - w_1| \leq 0.1 (0.5 e^2 - 2)(e^{0.2} - 1) = 0.03752$$

$$|y(0.4) - w_2| \leq 0.1 (0.5 e^2 - 2)(e^{0.2} - 1) = 0.08334$$

Higher-Order Taylor methods

Def:- The difference method

$w_0 = \alpha$

$w_{i+1} = w_i + h\,\phi(t_i, w_i)$, $\forall i = 0, 1, \dots, N-1$

has local trunction error

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\,\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i)$$

for each $i = 0, 1, \dots, N-1$, where y_i and y_{i+1} denote the solution of the differential eq. at t_i and t_{i+1} respectively

for example, Euler's method has local trunction error at the ith step

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) , \quad \forall i = 0, 1, \dots, N-1$$

Taylor's method of order n

$w_0 = \alpha$

$w_{i+1} = w_i + h\,T^{(n)}(t_i, w_i)$, $\forall i = 0, 1, \dots, N-1$

Where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h f'}{2}(t_i, w_i) + \dots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

Euler's method is Taylor's method of order one

Midpoint method :-

$w_0 = \alpha$

$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right), i = 0, 1, \dots, N-1$

Euler's modified method :-

$w_0 = \alpha$

$w_{i+1} = w_i + \frac{h}{2} f\left[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right]$

$i = 0, 1, \dots, N-1$

Range-Kutta 2nd order method

$w_0 = \alpha$

$w_{i+1} = w_i + hk_2$

$k_1 = f(t_i, w_i)$

$k_2 = f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} k_1\right)$

Fourth-order Runge-kutta method

The classical method is given by

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

with

$$k_1 = hf(t_i, w_i)$$

$$k_2 = hf(t_i + \frac{h}{2}, w_i + \frac{k_1}{2})$$

$$k_3 = hf(t_i + \frac{h}{2}, w_i + \frac{k_2}{2})$$

$$k_4 = hf(t_i + h, w + k_3)$$