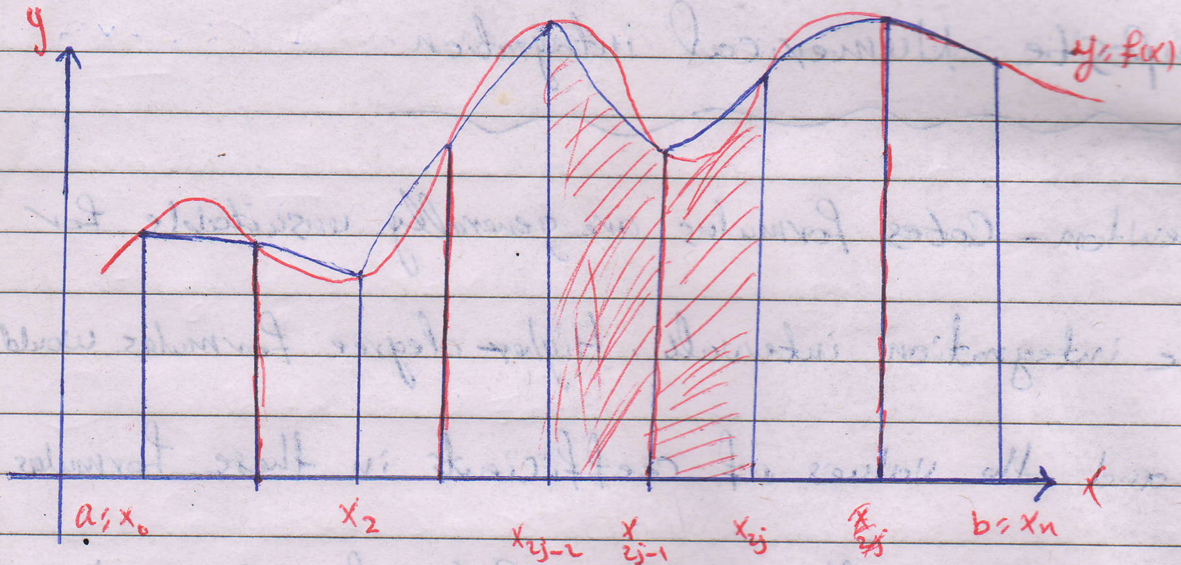
Composite Numerical integration

The Newton-Cotes formulas are generally unsuitable for use over large integration intervals. Higher degree formulas would be required and the values of coefficients in these formulas are difficult to obtain. Also Newton-Cotes formulas are based on interpolatory Polynomials that use equally spaced nodes, a procedure that is inaccurate over large intervals because of the oscillatory nature of high-degree polynomials.

Theorem:- Let $f \in C^4[a,b]$, n be even, $h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \dots, n$, there exists a $\mu \in (a,b)$ for which the Composite Simpson's rule for n subintervals can be written with its error term as

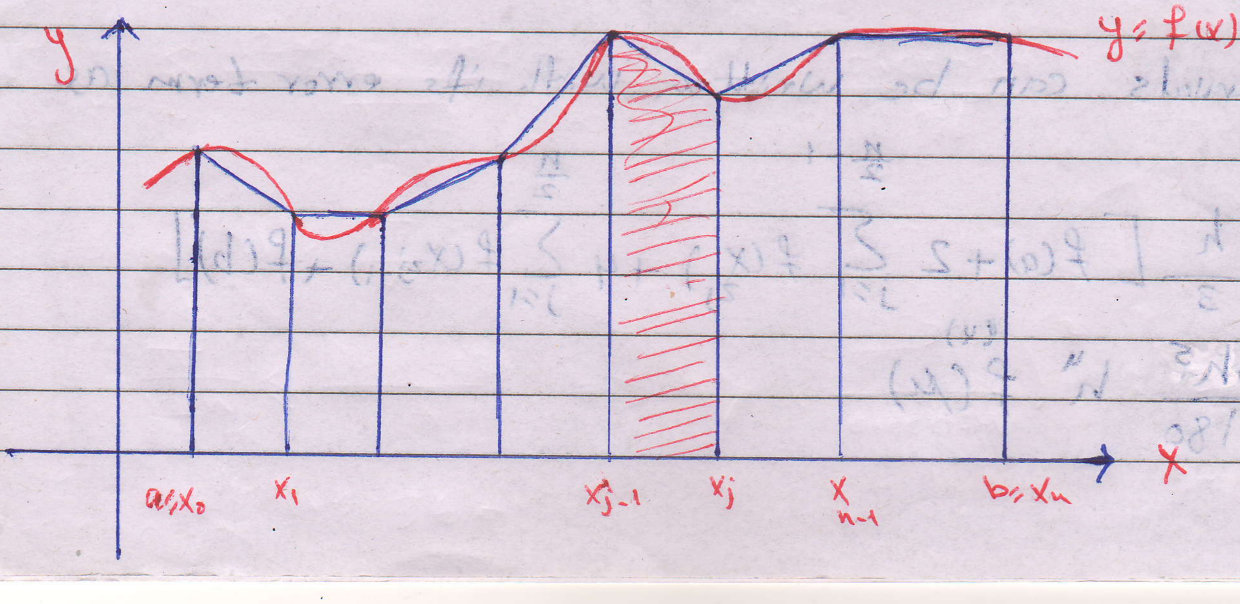
$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b)\right].$$

$$-\frac{b-a}{180} h^4 f^{(4)}(\mu)$$



Theorem :- Let $f \in C^2[a,b]$, $h = (b-a)/n$, and $x_j = a + jh$ for each $j = 0, 1, \ldots, n$. There exists a $\mu \in (a,b)$ for which the Composite Trapezoidale rule for n subintervals can be written with its error term as :

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12} h^2 f''(\mu)$$



Ex:- Use composite Simpson's rule to approximate

$\int_0^4 e^x dx$, let $n = 2, 4, 6$, compaire your results.

Sol :- Simpson rule on $[0,4]$,

$$\int_0^4 e^x dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.76958$$

The exact answer in this case is $e^4 - e^0 = 53.59$

and the error is -3.17143 is far large

applying Simpson's rule for $\int_0^2 e^x dx + \int_2^4 e^x dx$

we have

$$\int_0^4 e^x dx = \int_0^2 e^x dx + \int_2^4 e^x dx = \frac{1}{3}(e^0 + 4e^1 + e^2) + \frac{1}{3}(e^2 + 4e^3 + e^4)$$

$$= 53.86385$$

here the error has been reduce to -0.26570

ex:- Determine the values of h that will be ensure an approximation error of less than 0.00002 when approximating $\int_0^{\pi} \sin x\,dx$ and employing

(a) Composite Trapezoidal rule

(b) Composite Simpson's rule

Sol. The error term from the Composite Trapezoidal rule for $f(x) = \sin x$ on $[0, \pi]$

$$\left|\frac{\pi h^2}{12} f''(\mu)\right| = \left|\frac{\pi h^2}{12}(-\sin(\mu))\right| = \frac{\pi h^2}{12}|\sin(\mu)|$$

To ensure Sufficiently accuracy, with this technique we need have

$$\therefore \frac{\pi h^2}{12}|\sin(\mu)| \leq \frac{\pi h^2}{12} \leq 0.00002$$

$$h^2 \leq \frac{12 * 0.00002}{\pi} \Rightarrow h \leq \sqrt{\frac{0.00024}{\pi}}$$

since $h = \frac{\pi}{n}$

$$\therefore \frac{\pi^3}{12n} \leq \frac{0.00002}{1} \Rightarrow n \geq \left(\frac{\pi^3}{0.00024}\right)^{1/2} \simeq 359.44$$

$\therefore n = 360$

ملازمة رقم (٢)

Multiple integrals

Consider the double integrals

$$\iint_R f(x,y)\,dA$$

Where $R = \{(x,y) \mid a \le x \le b \;,\; c \le y \le d\}$ for some constants a, b, c, d is arectangular region in plan

* ~~illustration~~ : writting the double integral as iterated integral gives

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$$

let $k = \frac{d-c}{2}$, $h = \frac{b-a}{2}$, Then apply the composite Trapezoidal rule again to approximate the integral of this function of x

$$\int_c^d f(x,y)\,dy = \frac{k}{2}\left[f(x,c) + f(x,d) + 2f\left(x, \frac{c+d}{2}\right)\right]$$

$$\int_a^b \left(\int_c^d f(x,y)\,dy\right) dx \approx \int_a^b \left(\frac{d-c}{4}\right)\left[f(x,c) + 2f\left(x, \frac{c+d}{2}\right) + f(x,d)\right] dx$$

$$\approx \frac{b-a}{4}\left(\frac{d-c}{4}\right)\left[f(a,c)+2f\left(a,\frac{c+d}{2}\right)+f(a,d)\right]+\frac{b-a}{4}\left[2\left(\frac{d-c}{4}\right)\left\{f\left(\frac{a+b}{2},c\right)\right.\right.$$

$$\left.\left.+2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)\right\}\right)+\frac{b-a}{4}\left(2\frac{d-c}{4}\right)\left[f\left(\frac{a+b}{2},c\right)+\right.$$

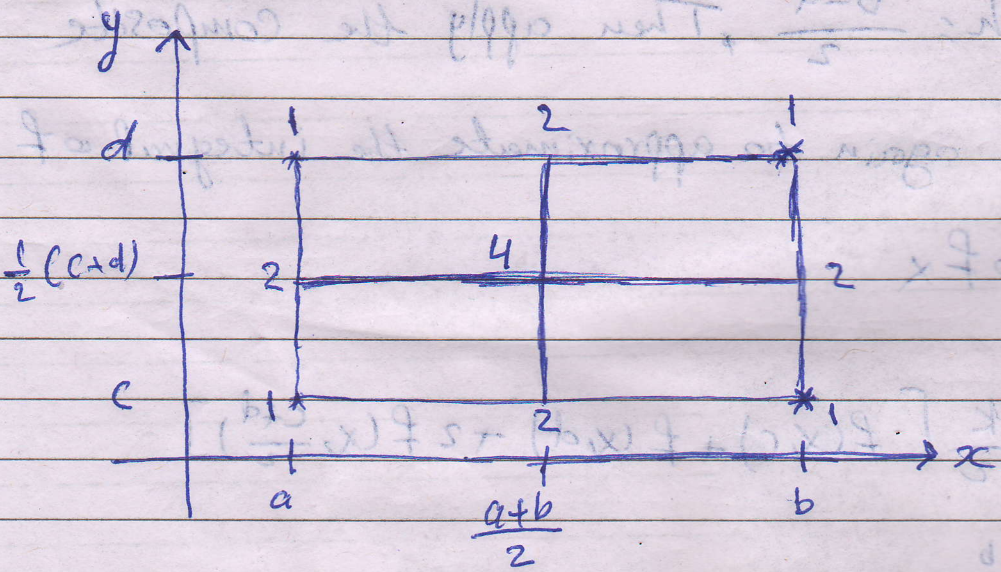
$$\left.2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+2f\left(\frac{a+b}{2},d\right)\right\}\Big)+\frac{b-a}{4}\left(\frac{d-c}{4}\right)\left[f(b,c)+\right.$$

$$\left.2f\left(b,\frac{c+d}{2}\right)+f(b,d)\right]$$

$$\approx \frac{(b-a)(d-c)}{16}\left[f(a,b)+f(a,d)+f(b,c)+f(b,d)+2\left\{f\left(\frac{a+b}{2},c\right)\right.\right.$$

$$\left.\left.+f\left(\frac{a+b}{2},d\right)+f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right\}+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]$$

This approximation is order $O\left((b-a)(d-c)\{(b-a)^2+(d-c)^2\}\right)$



Simpson's Rule

$$\int_a^b \int_c^d f(x,y)\,dy\,dx \approx \frac{hk}{9} \Big\{ \Big[f(x_0,y_0) + 2 \sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}, y_0)$$

$$+ 4 \sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}, y_0) + f(x_n, y_0) \Big] + 2 \Big[\sum_{j=1}^{\frac{m}{2}-1} f(x_0, y_{2j}) + \sum_{j=1}^{\frac{m}{2}-1} \sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}, y_{2j})$$

$$+ 4 \sum_{j=1}^{\frac{m}{2}-1} \sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}, y_{2j}) + \sum_{j=1}^{\frac{m}{2}-1} f(x_n, y_{2j}) \Big] + 4 \Big[\sum_{j=1}^{\frac{m}{2}} f(x_0, y_{2j-1}) +$$

$$2 \sum_{j=1}^{\frac{m}{2}} \sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}, y_{2j-1}) + 4 \sum_{j=1}^{\frac{m}{2}} \sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}, y_{2j-1})$$

$$+ \sum_{j=1}^{\frac{m}{2}} f(x_n, y_{2j-1}) \Big]$$

$$+ \Big[f(x_0, y_m) + 2 \sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}, y_m) + 4 \sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}, y_m) +$$

$$f(x_n, y_m) \Big]$$

and error term after Simplifying

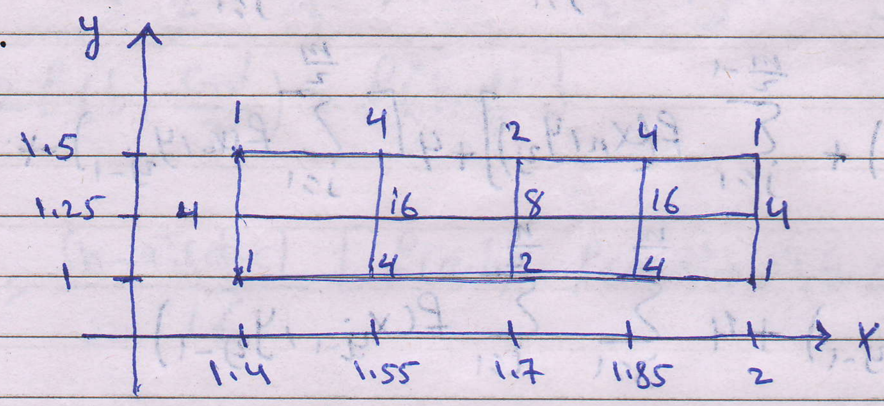
$$E = -\frac{(d-c)(b-a)}{180} \Big[h^4 \frac{\partial^4 f}{\partial x^4}(\bar{\eta}, \bar{\mu}) + k^4 \frac{\partial^4 f}{\partial y^4}(\hat{\eta}, \hat{\mu}) \Big]$$

for Some $(\bar{\eta}, \bar{\mu})$ and $(\hat{\eta}, \hat{\mu})$ in R

Ex:- Use composite Simpson's rule with $n=4$, $m=2$ to approximate

$$\int_{1.4}^{2}\int_{1}^{1.5} \ln(x+2y)\, dy\, dx$$

Sol:- first we construct the figure for Solution



here we find the step sizes for x and y

$$h = \frac{b-a}{n} \Rightarrow h = \frac{2-1.4}{4} = 1.5$$

$$k = \frac{d-c}{m} \Rightarrow k = \frac{1.5-1}{2} \Rightarrow k = 0.25$$

$$\int_{1.4}^{2}\int_{1}^{1.5} \ln(x_i+2y_j)\, dy\, dx \approx \frac{(0.15)(0.25)}{9} \sum_{i=0}^{4} \sum_{j=0}^{2} w_{ij} \ln(x_i+2$$

$$\approx 0.4295524387$$

We have $\frac{\partial^4 f}{\partial x^4}(x,y) = \frac{-6}{(x+2y)^4}$, $\frac{\partial^4 f}{\partial y^4}(x,y) = \frac{-96}{(x+2y)^4}$

$$|E| \leq \frac{(0.5)(0.6)}{180}\left[(0.15)^4 \max_{(x,y)\in R} \frac{6}{(x+2y)^4} + (0.25)^4 \max_{(x,y)\in R} \frac{96}{(x+2y)^4}\right]$$

$$\leq 4.72 \times 10^{-6}$$

The maximum values of the absolute values of these partial derivatives occure on R when $x=1.4$, $y=1.0$

The actual integral value to 10 decimal places is

$$\int_{1.4}^{2}\int_{1}^{1.5} \ln(x+2y)\,dy\,dx = 0.4295545265$$

So the approximation is accurate within

2.1×10^{-6}

Nonrectangular rigions

The use of approximations methods for double integrals is not limited to integrals with rectangular rigions of integration. The technique previously discussed can be modified to approximate double integrals of the form

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$

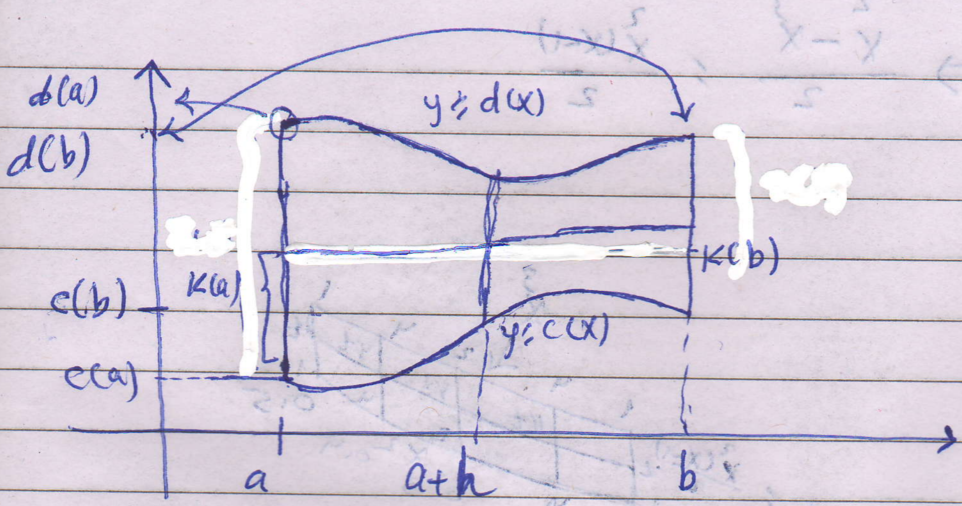
or

$$\int_c^d \int_{a(x)}^{b(x)} f(x,y)\,dx\,dy$$

We will use the basic Simpson's rule to integrate with respect to both variables. The step size for variable x is $h = \frac{b-a}{2}$, but for ~~y~~ ~~k(x)~~

y varies ~~x~~ with x

$$k(x) = \frac{d(x)-c(x)}{2}$$



$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx \approx \int_a^b \frac{k(x)}{3}\left[f(x,c(x)) + 4f(x, c(x)+k(x)) + f(x,d(x))\right]dx$$

$$\approx \frac{h}{3}\left\{\frac{k(a)}{3}\left[f(a,c(x)) + 4f(a, c(a)+k(a)) + f(a,d(a))\right]\right.$$

$$+4\frac{k(a+h)}{3}\left[f(a+h, c(a+h)) + 4f(a+h, c(a+h)+k(a+h))\right.$$

$$\left.+ f(a+h, d(a+h))\right] + \frac{k(b)}{3}\left[f(b,c(b)) + 4f(b, c(b)+k(b))\right.$$

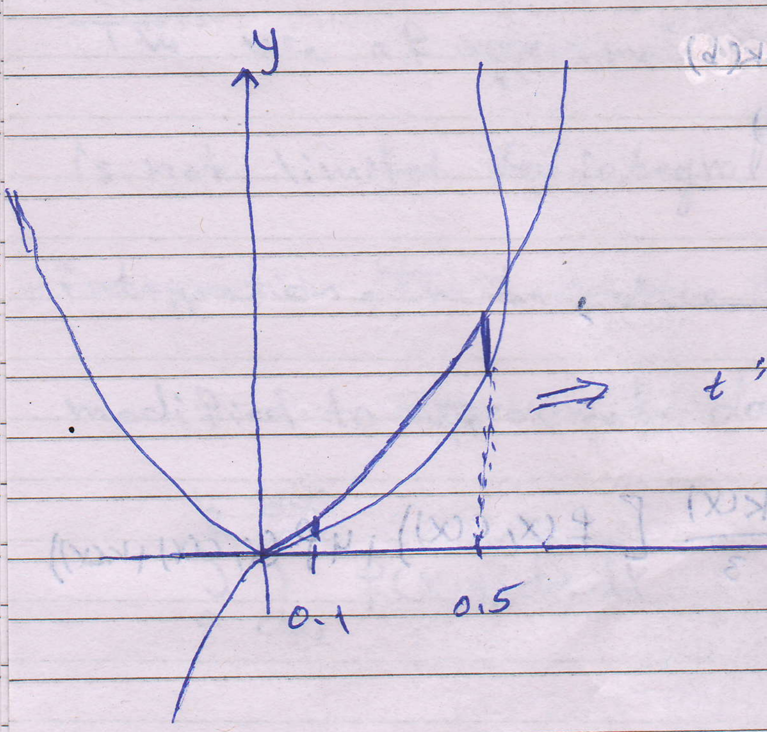
$$\left.\left.+ f(b,d(b))\right]\right\}$$

Ex:- Solve the integration Numerically

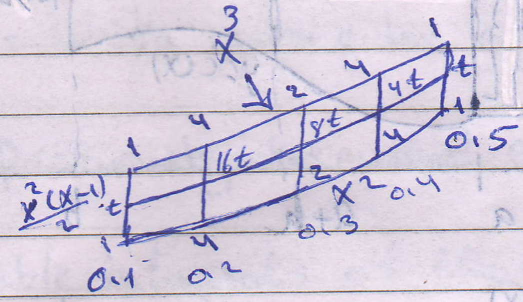
$$\int_{0.1}^{0.5}\int_{x^3}^{x^2} e^{y/x}\,dy\,dx \quad , n = 4 \; , m = 2$$

Solution $h = \frac{0.5-0.1}{4} \Rightarrow h = \frac{0.4}{4} = 0.1$

$$k(x) = \frac{d(x) - c(x)}{2} \Rightarrow \frac{x^2 - x^3}{2} = \frac{x^2(x-1)}{2}$$



$\Rightarrow$ $t = \frac{x^2(x-1)}{2}$



Triple integral approximation

Triple integrals of the form

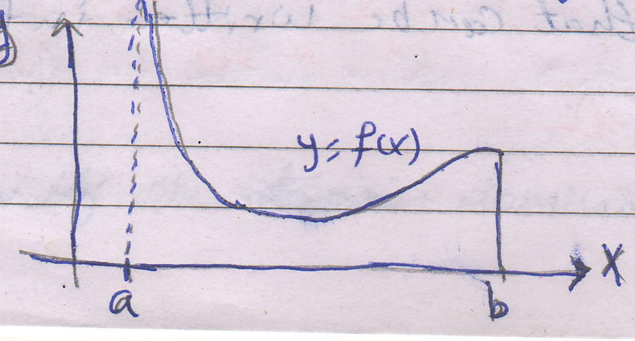
$$\int_a^b \int_{c(x)}^{d(x)} \int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Improper integrals:

Improper integrals result when the notation of integration is extended to an interval of integration on which the function is unbounded or to an interval with one or more infinite endpoints.

1 – Left endpoint singularity

We say that f has a singularity at the endpoint a



It is shown in calculus that the improper integral with singularity at the left endpoint

$$\int_a^b \frac{dx}{(x-a)^p}$$, converges iff $0<p<1$, and in th

Case $$\int_a^b \frac{1}{(x-a)^p}\,dx = \lim_{M\to a^+} \frac{(x-a)^{1-p}}{1-p}\Big|_{x=M}^{x=b}$$

$$= \frac{(b-a)^{1-p}}{1-p}$$

ex:- show that the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ conv. but $\int_0^1 \frac{1}{x^2}\,dx$ is div.

Sol $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{M\to 0^+} \int_M^1 x^{-1/2}\,dx = \lim_{M\to 0} 2x^{1/2}\Big|_M^1 = 2-0=$$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{M\to 0^+} \int_M^1 x^{-2}\,dx = \lim_{M\to 0^+} -x^{-1}\Big|_M^{+1}$$ is unbou

If f is a function that can be written in the form

$$f(x) = \frac{g(x)}{(x-a)^p}$$

Where $0<p<1$ and g is continuous on $[a,b]$, then

improper integral $\int_a^b f(x)\,dx$

also exists. We will approximate this integral using

Composite Simpson's rule provided that $g \in C^5[a,b]$.

In this case, we can construct the fourth Taylor

Polynomial, $P_4(x)$ for g about a

$$P_4(x) = g(a) + g'(a)(x-a) + g''(a)\frac{(x-a)^2}{2!} + \frac{g'''(a)}{3!}(x-a)^3 + \frac{g^{(4)}(a)}{4!}(x-a)^4,$$

and Write

$$\int_a^b f(x)\,dx = \int_a^b \frac{g(x)-P_4(x)}{(x-a)^p}\,dx + \int_a^b \frac{P_4(x)}{(x-a)^p}\,dx$$

because $P_4(x)$ is a Polynomial, we can exactly determine

the value of

$$\int_a^b \frac{P_4(x)}{(x-a)^p}\,dx = \sum_{k=0}^{4} \int_a^b \frac{g^{(k)}(a)}{k!}(x-a)^{k-p}\,dx = \sum_{k=0}^{4} \frac{g^{(k)}(a)}{k!\,(k+1-p)}(b-a)^{k+1-p} \quad (*)$$

This generally the ~~domain~~ dominant portion of the approximation, especially when the Taylor Polynomial $P_4(x)$ agrees closely with $g(x)$ throughout the interval $[a,b]$, To approximate the integral of f

To approximate integral of f, we must add to this value the approximation of

$$\int_a^b \frac{g(x)-P_4(x)}{(x-a)^p}\,dx$$

To determine this, we first define

$$G(x) = \begin{cases} \dfrac{g(x)-P_4(x)}{(x-a)^p} & \text{if } a < x \leq b \\ 0 & \text{if } x = a \end{cases}$$

This gives us a continuous function on $[a,b]$, In fa
$0<p<1$ and $P_4^{(k)}(a)$ agrees with $g^{(k)}(a)$ for each $k=0,1,$
So, we have $G \in C^4[a,b]$, This implies that the Compos
Simpson's rule can be applied to approximate the integra
of G on $[a,b]$, adding this approximation to the value in
eq. $(*)$, gives an approximation to the improper integra
of f on $[a,b]$

Ex:- Use the composite Simpson's rule with $h = 0.25$

$$\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$$

Solution

The fourth Taylor Polynomials for e^x

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

So the dominant portion of the approximation to $\int_0^1 e^x\,dx$

~~about x=0~~ is

$$\int_0^1 \frac{P_4(x)}{\sqrt{x}}\,dx = \int_0^1 \left(x^{-1/2} + x^{1/2} + \frac{1}{2}x^{3/2} + \frac{1}{6}x^{5/2} + \frac{1}{24}x^{7/2} \right) dx$$

$$= \lim_{M \to 0^+} \left[2x^{1/2} + \frac{2}{3}x^{3/2} + \frac{1}{5}x^{5/2} + \frac{1}{21}x^{7/2} + \frac{1}{108}x^{9/2} \right]_M^1$$

$$= 2 + \frac{2}{3} + \frac{1}{5} + \frac{1}{21} + \frac{1}{108} \approx 2.9235450$$

for the second Portion of the approximation to

$\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$, We need to approximate $\int_0^1 G(x)\,dx$ where

$$G(x) = \begin{cases} \frac{1}{\sqrt{x}}\left(e^x - P_4(x)\right) & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$$

$$\int_0^1 G(x)dx \approx \frac{0.25}{3}\left[0+4(0.000017)+2(0.0004013)+4(0.0026026\right.$$

$$\left.+0.0099485\right]$$

$$\approx 0.0017691$$

Hence

$$\int_0^1 \frac{e^x}{\sqrt{x}}dx \approx 2.9235450 + 0.0017691$$

$$\approx 2.9253141$$

This result is accurate to with in the accuracy of the composite Simpson's rule approximation for the function G, Because $|G^{4}(x)| < 1$ on $[0,1]$. the error bounded by

$$\frac{1-0}{180}(0.25)^4 \approx 0.0000217$$

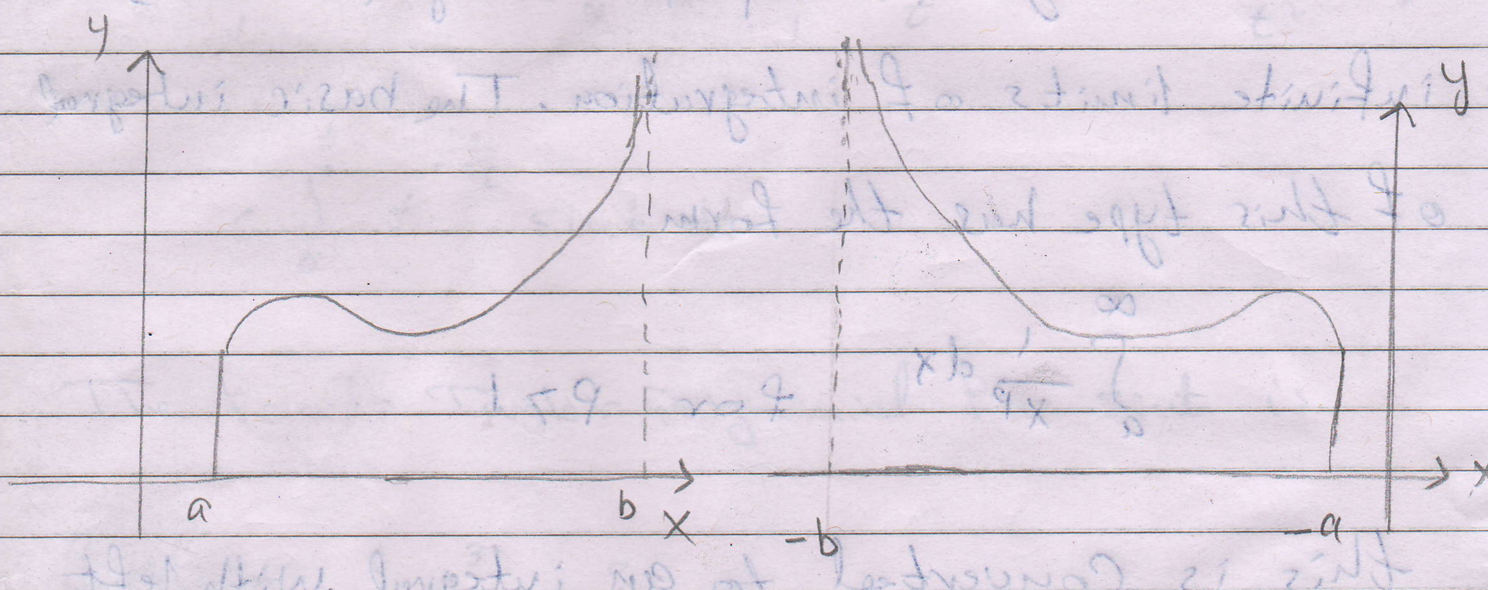
Right - End point Singularity

To approximate the improper integral with singularity at the right endpoint b instead of the left endpoint a. Alternatively we can make the substitution

$$z = -x \quad , dz = -dx$$

to change the improper integral into one of the form

$$\int_a^b f(x)dx = \int_{-b}^{-a} f(-z)\,dz$$



An improper integral with a singularity at c where $a<c<b$, is treated as the ~~[illegible]~~ sum of improper integrals with endpoint singularities since

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Infinity Singularity

The other type of improper integral involves infinite limits of integration. The basic integral of this type has the form

$$\int_a^\infty \frac{1}{x^p}\,dx \quad \text{for } p>1$$

this is converted to an integral with left end-point singularity at 0 by making the integration substitution

$t = x^{-1}$, $dt = -x^{-2}dx$, so $dx = -x^{2}dt = -t^{2}dt$

then

$$\int_a^{\infty} \frac{1}{x^p}\,dx = \int_{1/a}^{0} -\frac{t^p}{t^2}\,dt = \int_0^{1/a} \frac{1}{t^{2-p}}\,dt.$$

Ex :- Approximate the value of

$$I = \int_1^{\infty} x^{-\frac{3}{2}} \sin\left(\frac{1}{x}\right)dx$$

Sol $dt = -x^{-2}dx$, so $dx = -x^{2}dt = -t^{-2}dt$

$$I = \int_1^{\infty} x^{-\frac{3}{2}} \sin\left(\frac{1}{x}\right)dx = \int_1^{0} \left(\frac{1}{t}\right)^{-\frac{3}{2}} \sin t \cdot \left(\frac{-1}{t^2}\,dt\right)$$

$$= \int_0^{1} t^{-\frac{1}{2}} \sin t\,dt$$

The fourth Taylor polynomial for $\sin t$ is

$P_4(t) = t - \frac{1}{6}t^3$, so

$$G(t) = \begin{cases} \dfrac{\sin t - t + \frac{1}{6}t^3}{} & 0 < t \leq 1 \\ 0 & t = 0 \end{cases}$$

is in $C^4[0,1]$, and we have

$$I = \int_0^1 t^{-\frac{1}{2}}\left[t - \frac{1}{6}t^3\right) dt + \int_0^1 \frac{\sin t - t + \frac{1}{6}t^3}{t^{1/2}} dt$$

$$= \left[\frac{2}{3}t^{\frac{3}{2}} - \frac{1}{21}t^{\frac{7}{2}}\right]_0^1 + \int_0^1 \frac{\sin t - t + \frac{1}{6}t^3}{t^{1/2}} dt$$

$$= 0.61904761 + \underbrace{\int_0^1 \frac{\sin t - t + \frac{1}{6}t^3}{t^{1/2}} dt}_{\text{Composite simpson rule}}$$

$$= 0.61904761 + 0.0014890097$$

$$= 0.62053661$$