Electricity and Magnetics II

Lecture No.(13)- Semester 2

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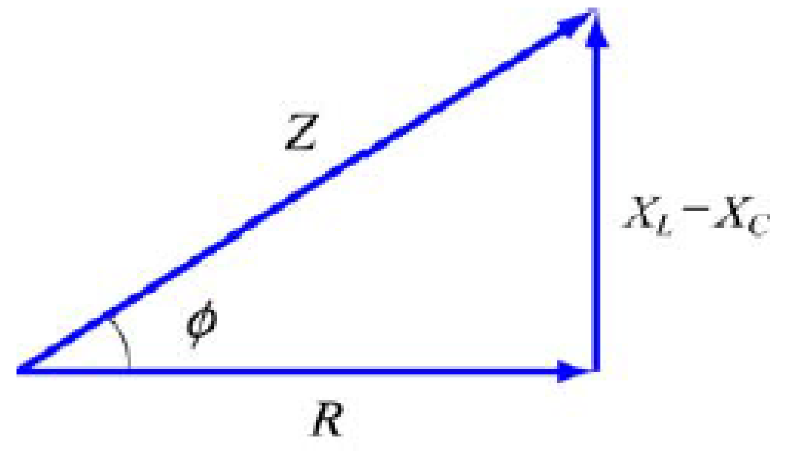
12.3.1 Impedance

We have already seen that the inductive reactance $X_L = \omega L$ and capacitance reactance $X_C = 1/\omega C$ play the role of an effective resistance in the purely inductive and capacitive circuits, respectively. In the series *RLC* circuit, the effective resistance is the *impedance*, defined as

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad (12.3.15)$$

The relationship between Z, X_L and X_C can be represented by the diagram shown in Figure 12.3.4:



The impedance also has SI units of ohms. In terms of Z, the current may be rewritten as

$$I(t) = \frac{V_0}{Z}\sin(\omega t - \phi) \qquad (12.3.16)$$

Notice that the impedance Z also depends on the angular frequency ω, as do X_L and X_C.

Using Eq. (12.3.6) for the phase ϕ and Eq. (12.3.15) for Z, we may readily recover the limits for simple circuit (with only one element). A summary is provided in Table 12.1 below:

Simple Circuit	R	L	C	$X_L = \omega L$	$X_C = \dfrac{1}{\omega C}$	$\phi = \tan^{-1}\left(\dfrac{X_L - X_C}{R}\right)$	$Z = \sqrt{R^2 + (X_L - X_C)^2}$
purely resistive	R	0	∞	0	0	0	R
purely inductive	0	L	∞	X_L	0	$\pi/2$	X_L
purely capacitive	0	0	C	0	X_C	$-\pi/2$	X_C

12.3.2 Resonance

Eq. (12.3.15) indicates that the amplitude of the current $I_0 = V_0 / Z$ reaches a maximum when Z is at a minimum. This occurs when $X_L = X_C$, or $\omega L = 1/\omega C$, leading to

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{12.3.17}$$

The phenomenon at which I_0 reaches a maximum is called a resonance, and the frequency ω_0 is called the resonant frequency. At resonance, the impedance becomes $Z = R$, the amplitude of the current is

$$I_0 = \frac{V_0}{R} \tag{12.3.18}$$

ınd the phase is

$$\phi = 0 \tag{12.3.19}$$

as can be seen from Eq. (12.3.5). The qualitative behavior is illustrated in Figure 12.3.5.

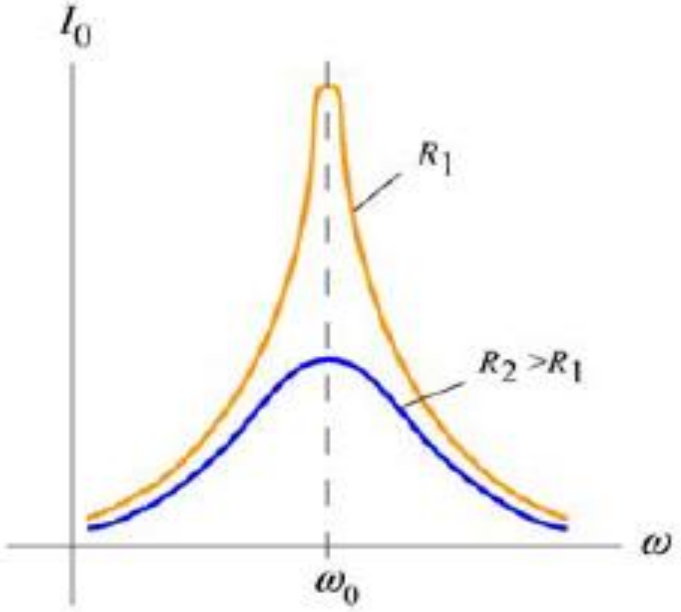


Figure 12.3.5 The amplitude of the current as a function of ω in the driven *RLC* circuit.

12.4 Power in an AC circuit

In the series *RLC* circuit, the instantaneous power delivered by the AC generator is given by

$$\begin{aligned} P(t) = I(t)V(t) &= \frac{V_0}{Z}\sin(\omega t - \phi)\cdot V_0 \sin\omega t = \frac{V_0^2}{Z}\sin(\omega t - \phi)\sin\omega t \\ &= \frac{V_0^2}{Z}\left(\sin^2\omega t\cos\phi - \sin\omega t\cos\omega t\sin\phi\right) \end{aligned} \tag{12.4.1}$$

where we have used the trigonometric identity

$$\sin(\omega t - \phi) = \sin\omega t\cos\phi - \cos\omega t\sin\phi \tag{12.4.2}$$

The time average of the power is

$$\begin{aligned}\langle P(t)\rangle &= \frac{1}{T}\int_0^T \frac{V_0^{\,2}}{Z}\sin^2\omega t\cos\phi\, dt - \frac{1}{T}\int_0^T \frac{V_0^{\,2}}{Z}\sin\omega t\cos\omega t\sin\phi\, dt \\ &= \frac{V_0^{\,2}}{Z}\cos\phi\langle\sin^2\omega t\rangle - \frac{V_0^{\,2}}{Z}\sin\phi\langle\sin\omega t\cos\omega t\rangle \\ &= \frac{1}{2}\frac{V_0^{\,2}}{Z}\cos\phi\end{aligned} \tag{12.4.3}$$

where Eqs. (12.2.5) and (12.2.7) have been used. In terms of the rms quantities, the average power can be rewritten as

$$\langle P(t)\rangle = \frac{1}{2}\frac{V_0^{\,2}}{Z}\cos\phi = \frac{V_{\text{rms}}^{\;\;2}}{Z}\cos\phi = I_{\text{rms}}V_{\text{rms}}\cos\phi \tag{12.4.4}$$

The quantity $\cos\phi$ is called the *power factor*. From Figure 12.3.4, one can readily show that

$$\cos\phi = \frac{R}{Z} \tag{12.4.5}$$

Thus, we may rewrite $\langle P(t)\rangle$ as

$$\langle P(t)\rangle = I_{\rm rms}V_{\rm rms}\left(\frac{R}{Z}\right) = I_{\rm rms}\left(\frac{V_{\rm rms}}{Z}\right)R = I_{\rm rms}^2 R \qquad (12.4.6)$$

In Figure 12.4.1, we plot the average power as a function of the driving angular frequency ω.

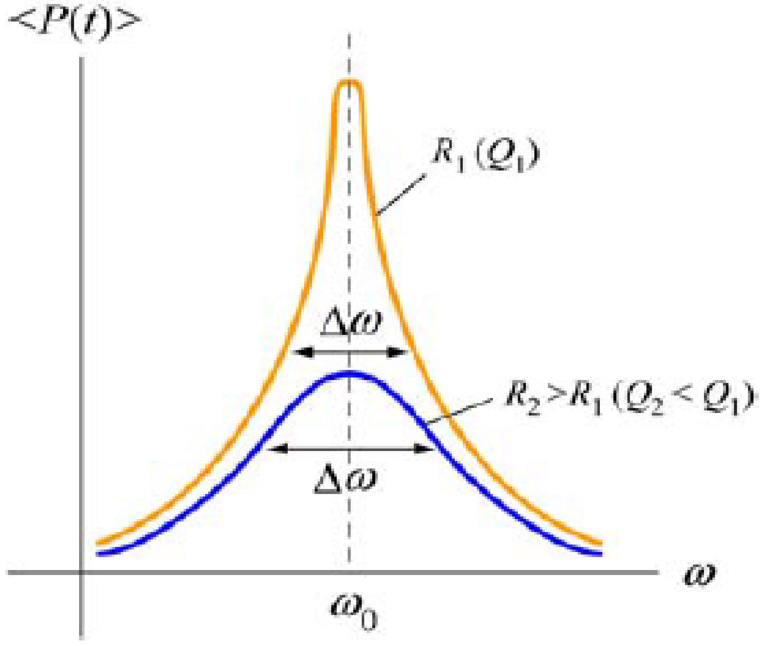


Figure 12.4.1 Average power as a function of frequency in a driven series *RLC* circuit.

We see that $\langle P(t)\rangle$ attains the maximum when $\cos\phi = 1$, or $Z = R$, which is the resonance condition. At resonance, we have

$$\langle P\rangle_{\max} = I_{\mathrm{rms}}V_{\mathrm{rms}} = \frac{V_{\mathrm{rms}}^2}{R} \tag{12.4.7}$$

12.4.1 Width of the Peak

The peak has a line width. One way to characterize the width is to define $\Delta\omega = \omega_+ - \omega_-$, where $\omega_\pm$ are the values of the driving angular frequency such that the power is equal to half its maximum power at resonance. This is called *full width at half maximum*, as illustrated in Figure 12.4.2. The width $\Delta\omega$ increases with resistance R.

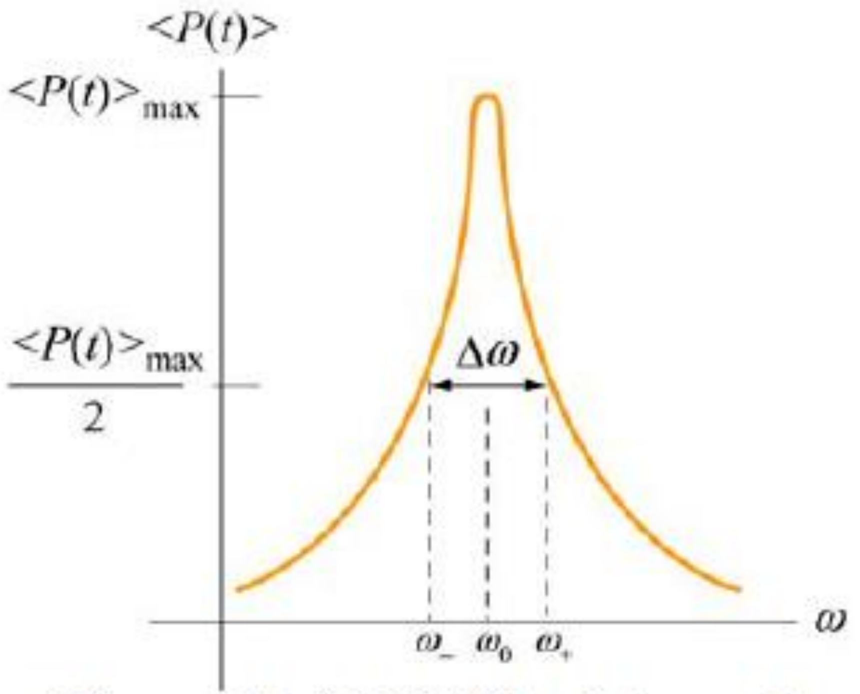


Figure 12.4.2 Width of the peak

To find $\Delta\omega$, it is instructive to first rewrite the average power $\langle P(t)\rangle$ as

$$\langle P(t)\rangle = \frac{1}{2}\frac{V_0^2 R}{R^2 + (\omega L - 1/\omega C)^2} = \frac{1}{2}\frac{V_0^2 R\omega^2}{\omega^2 R^2 + L^2(\omega^2 - \omega_0^2)^2} \tag{12.4.8}$$

with $\langle P(t)\rangle_{\max} = V_0^2 / 2R$. The condition for finding $\omega_\pm$ is

$$\frac{1}{2}\langle P(t)\rangle_{\max} = \langle P(t)\rangle\Big|_{\omega_\pm} \quad \Rightarrow \quad = \frac{V_0^2}{4R} = \frac{1}{2}\frac{V_0^2 R\omega^2}{\omega^2 R^2 + L^2(\omega^2 - \omega_0^2)^2}\Bigg|_{\omega_\pm} \tag{12.4.9}$$

which gives

$$(\omega^2 - \omega_0^2)^2 = \left(\frac{R\omega}{L}\right)^2 \tag{12.4.10}$$

Taking square roots yields two solutions, which we analyze separately.

case 1: Taking the positive root leads to

$$\omega_+^2 - \omega_0^2 = +\frac{R\omega_+}{L} \qquad (12.4.11)$$

Solving the quadratic equation, the solution with positive root is

$$\omega_+ = \frac{R}{2L} + \sqrt{\left(\frac{R}{4L}\right)^2 + \omega_0^{\,2}} \qquad (12.4.12)$$

Case 2: Taking the negative root of Eq. (12.4.10) gives

$$\omega_-^2 - \omega_0^2 = -\frac{R\omega_-}{L} \tag{12.4.13}$$

The solution to this quadratic equation with positive root is

$$\omega_- = -\frac{R}{2L} + \sqrt{\left(\frac{R}{4L}\right)^2 + \omega_0{}^2} \tag{12.4.14}$$

The width at half maximum is then

$$\Delta\omega = \omega_+ - \omega_- = \frac{R}{L} \tag{12.4.15}$$

Once the width $\Delta\omega$ is known, the quality factor Q (not to be confused with charge) can be obtained as

$$\boxed{Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R}} \tag{12.4.16}$$

Comparing the above equation with Eq. (11.8.17), we see that both expressions agree with each other in the limit where the resistance is small, and $\omega' = \sqrt{\omega_0^2 - (R/2L)^2} \approx \omega_0$.

12.5 Transformer

A transformer is a device used to increase or decrease the AC voltage in a circuit. A typical device consists of two coils of wire, a primary and a secondary, wound around an iron core, as illustrated in Figure 12.5.1. The primary coil, with N_1 turns, is connected to alternating voltage source $V_1(t)$. The secondary coil has N_2 turns and is connected to a "load resistance" R_2. The way transformers operate is based on the principle that an alternating current in the primary coil will induce an alternating emf on the secondary coil due to their mutual inductance.

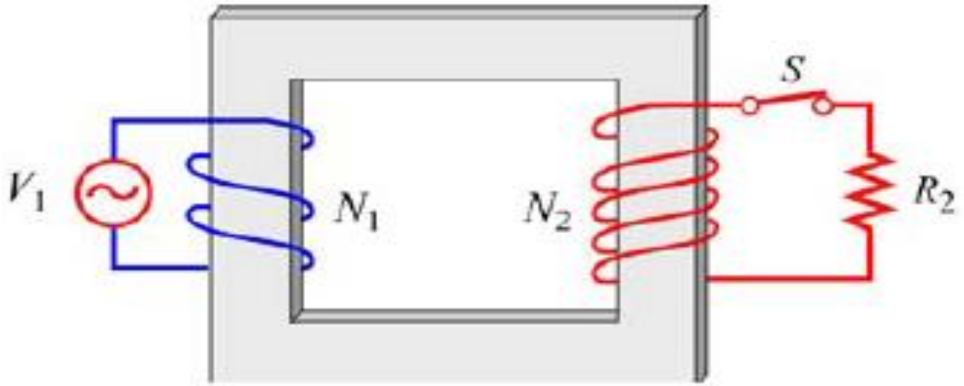


Figure 12.5.1 A transformer

In the primary circuit, neglecting the small resistance in the coil, Faraday's law of induction implies

$$V_1 = -N_1 \frac{d\Phi_B}{dt} \tag{12.5.1}$$

where Φ_B is the magnetic flux through one turn of the primary coil. The iron core, which extends from the primary to the secondary coils, serves to increase the magnetic field produced by the current in the primary coil and ensure that nearly all the magnetic flux through the primary coil also passes through each turn of the secondary coil. Thus, the voltage (or induced emf) across the secondary coil is

$$V_2 = -N_2 \frac{d\Phi_B}{dt} \tag{12.5.2}$$

In the case of an ideal transformer, power loss due to Joule heating can be ignored, so that the power supplied by the primary coil is completely transferred to the secondary coil:

$$I_1 V_1 = I_2 V_2 \tag{12.5.3}$$

In addition, no magnetic flux leaks out from the iron core, and the flux Φ_B through each turn is the same in both the primary and the secondary coils. Combining the two expressions, we are lead to the transformer equation:

$$\boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}} \tag{12.5.4}$$

By combining the two equations above, the transformation of currents in the two coils may be obtained as:

$$I_1 = \left(\frac{V_2}{V_1}\right) I_2 = \left(\frac{N_2}{N_1}\right) I_2 \tag{12.5.5}$$

Thus, we see that the ratio of the output voltage to the input voltage is determined by the *turn ratio* N_2 / N_1. If $N_2 > N_1$, then $V_2 > V_1$, which means that the output voltage in the second coil is greater than the input voltage in the primary coil. A transformer with $N_2 > N_1$ is called a *step-up* transformer. On the other hand, if $N_2 < N_1$, then $V_2 < V_1$, and the output voltage is smaller than the input. A transformer with $N_2 < N_1$ is called a *step-down* transformer.

A series RLC circuit with $L = 160\ \text{mH}$, $C = 100\ \mu\text{F}$, and $R = 40.0\ \Omega$ is connected to a sinusoidal voltage $V(t) = (40.0\ \text{V})\sin\omega t$, with $\omega = 200\ \text{rad/s}$.

(a) What is the impedance of the circuit?

(b) Let the current at any instant in the circuit be $I(t) = I_0 \sin(\omega t - \phi)$. Find I_0.

(c) What is the phase ϕ?

Solution:

(a) The impedance of a series RLC circuit is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{12.9.1}$$

where

$$X_L = \omega L \tag{12.9.2}$$

and

$$X_C = \frac{1}{\omega C} \tag{12.9.3}$$

are the inductive reactance and the capacitive reactance, respectively. Since the general expression of the voltage source is $V(t) = V_0 \sin(\omega t)$, where V_0 is the maximum output voltage and ω is the angular frequency, we have $V_0 = 40$ V and $\omega = 200$ rad/s. Thus, the impedance Z becomes

$$Z = \sqrt{(40.0\ \Omega)^2 + \left((200\ \text{rad/s})(0.160\ \text{H}) - \frac{1}{(200\ \text{rad/s})(100 \times 10^{-6}\ \text{F})}\right)^2} \qquad (12.9.4)$$

$$= 43.9\,\Omega$$

(b) With $V_0 = 40.0\,\text{V}$, the amplitude of the current is given by

$$I_0 = \frac{V_0}{Z} = \frac{40.0\,\text{V}}{43.9\,\Omega} = 0.911\,\text{A} \qquad (12.9.5)$$

(c) The phase between the current and the voltage is determined by

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{\omega L - \dfrac{1}{\omega C}}{R}\right)$$

$$= \tan^{-1}\left(\frac{(200\text{ rad/s})(0.160\text{ H}) - \dfrac{1}{(200\text{ rad/s})(100\times10^{-6}\text{ F})}}{40.0\ \Omega}\right) = -24.2^\circ \qquad (12.9.6)$$

Suppose an AC generator with $V(t) = (150\text{ V})\sin(100t)$ is connected to a series RLC circuit with $R = 40.0\ \Omega$, $L = 80.0$ mH, and $C = 50.0\ \mu\text{F}$, as shown in Figure 12.9.1.

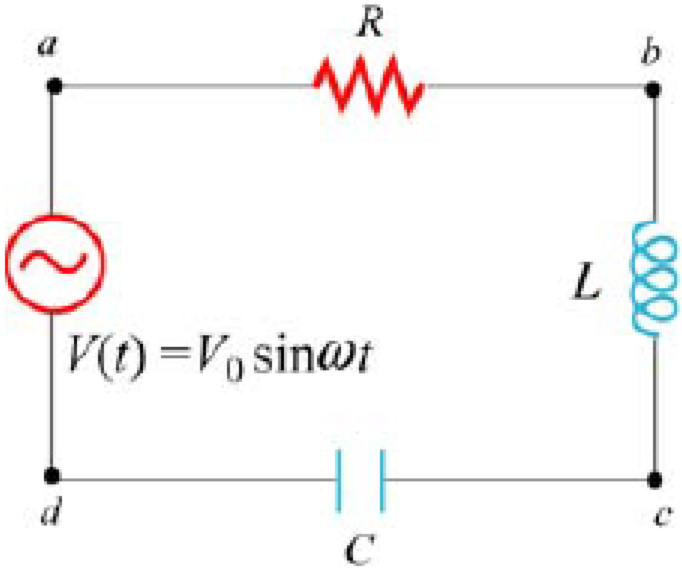


Figure 12.9.1 *RLC* series circuit

(a) Calculate V_{R0}, V_{L0} and V_{C0}, the maximum of the voltage drops across each circuit element.

(b) Calculate the maximum potential difference across the inductor and the capacitor between points b and d shown in Figure 12.9.1.